Discrete Mathematics 339 (2016) 2345–2356

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Split graphs and Nordhaus–Gaddum graphs

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ARTICLE INFO

Article history:
Received 11 June 2015
Received in revised form 30 March 2016
Accepted 2 April 2016
Available online 6 May 2016

Keywords:
Nordhaus–Gaddum theorem
NG-graphs
Split graphs
Pseudo-split graphs
Degree sequence characterization

ABSTRACT

A graph G is an NG-graph if $\chi(G) + \chi(\overline{G}) = |V(G)| + 1$. We characterize NG-graphs solely from degree sequences leading to a linear-time recognition algorithm. We also explore the connections between NG-graphs and split graphs. There are three types of NG-graphs and split graphs can also be divided naturally into two categories, balanced and unbalanced. We characterize each of these five classes by degree sequence. We construct bijections between classes of NG-graphs and balanced and unbalanced split graphs which, together with the known formula for the number of split graphs on n vertices, allows us to compute the sizes of each of these classes. Finally, we provide a bijection between unbalanced split graphs on n vertices and split graphs on $n - 1$ or fewer vertices providing evidence for our conjecture that the rapid growth in the number of split graphs comes from the balanced split graphs.

1. Introduction

For a graph G, the number of vertices in a largest clique in G is denoted by $\omega(G)$ and the number of vertices in a largest stable set (independent set) in G is denoted by $\alpha(G)$. We denote the complement of G by $\overline{G}$ and the graph induced in G by $X \in V(G)$ by $G[X]$. We write $nbh(x)$ to denote the set of vertices adjacent to vertex x. For a set of graphs, $\mathcal{C}$, we denote by $\mathcal{C}_n$ the set of graphs in $\mathcal{C}$ with n vertices.

A well-known theorem by Nordhaus and Gaddum [14] states that the following is true for any graph G:

$$2\sqrt{|V(G)|} \leq \chi(G) + \chi(\bar{G}) \leq |V(G)| + 1.$$

We call G a *Nordhaus–Gaddum graph* or *NG-graph* if $\chi(G) + \chi(\bar{G}) = |V(G)| + 1$. Finck [9] and Starr and Turner [16] provide two different characterizations of NG-graphs. More recently, Collins and Trenk [7] define the *ABC*-partition of a graph and characterize NG-graphs in terms of this partition.

Definition 1. For a graph G, the *ABC-partition* of $V(G)$ (or of G) is
$A_G = \{v \in V(G) : deg(v) = \chi(G) - 1\}$
$B_G = \{v \in V(G) : deg(v) > \chi(G) - 1\}$
$C_G = \{v \in V(G) : deg(v) < \chi(G) - 1\}$.
When it is unambiguous, we write $A = A_G$, $B = B_G$, $C = C_G$.

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http://dx.doi.org/10.1016/j.disc.2016.04.001
0012-365X/

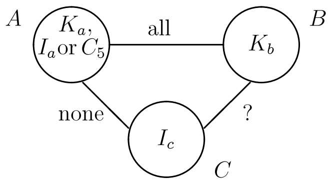

Fig. 1. The forms of an NG-graph.

Theorem 2 (*Collins and Trenk [7]*). *A graph G is an NG-graph if and only if its ABC-partition satisfies*
(i) *$A \neq \emptyset$ and $G[A]$ is a clique, a stable set, or a 5-cycle*
(ii) *$G[B]$ is a clique*
(iii) *$G[C]$ is a stable set*
(iv) *$uv \in E(G)$ for all $u \in A$, $v \in B$*
(v) *$uw \notin E(G)$ for all $u \in A$, $w \in C$.*

By (i) of Theorem 2, there are three possible forms of an NG-graph. (See Fig. 1.)

Definition 3. We say that G is an NG-1 graph if $G[A]$ is a clique, an NG-2graph if $G[A]$ is a stable set, and an NG-3 graph if $G[A]$ is a 5-cycle. We also let $\mathcal{NG}$-1 be the set of NG-1 graphs and likewise define the sets $\mathcal{NG}$-2 and $\mathcal{NG}$-3.

Observe that a graph G is in both $\mathcal{NG}$-1 and $\mathcal{NG}$-2 if and only if $|A_G| = 1$. The characterization in Theorem 2 not only provides a clear description of NG-graphs but it also lends itself to an $O(|V(G)|^3)$-time recognition algorithm for NG-graphs [7]. More importantly for this article, it shows that NG-graphs are related to split graphs and pseudo-split graphs.

A *split graph* is a graph G whose vertex set can be partitioned as $V(G) = K \cup S$, where K induces a clique and S induces a stable set in G. A detailed introduction to this class appears in [11]. Split graphs are a well-known class of perfect graphs, and thus $\chi(G) = \omega(G)$ for split graphs. Split graphs also have elegant characterization theorems. Földes and Hammer [10] give a forbidden subgraph characterization of split graphs as those graphs with no induced $2K_2$, C_4 or C_5. Split graphs also have a degree sequence characterization due to Hammer and Simeone [12] which we present in Theorem 13. This latter characterization implies that split graphs can be recognized in linear time.

Blázsik et al. [3] consider the class of graphs that do not contain C_4 and $2K_2$ as induced subgraphs, later referred to as *pseudo-split graphs* [13]. They show that like split graphs, pseudo-split graphs can be defined in terms of vertex sets partitions. In particular, a graph G is a pseudo-split graph if and only if $V(G)$ can be partitioned into three parts so that (i) first part is either empty or induces a 5-cycle, the second part a clique, the third part a stable set and (ii) whenever the first part is a 5-cycle, every vertex in the first part is adjacent to every vertex in the second part but there are no edges between the first part and the third part. Interestingly, Blázsik et al. also note that pseudo-split graphs are almost extremal in terms of the Nordhaus–Gaddum inequality because for any such graph G, $\chi(G) + \chi(\bar{G}) \geq |V(G)|$. In the process of proving this result, they show that if G contains an induced 5-cycle then $\chi(G) + \chi(\bar{G}) \geq |V(G)| + 1$ and thus G is an NG-graph.

The next result follows from Theorem 2 and the characterization of pseudo-split graphs discussed above. In particular, a graph is an NG-3 graph if and only if it is a pseudo-split graph containing an induced 5-cycle.

Remark 4. A graph is a pseudo-split graph if and only if it is a split graph or an NG-3 graph.

Proposition 5. *Let G be an NG-graph. Then G is a split graph if and only if $G \in (\mathcal{NG}\text{-}1 \cup \mathcal{NG}\text{-}2)$.*

Proof. By definition, NG-3 graphs contain an induced C_5, hence are not split graphs. Now suppose $G \in (\mathcal{NG}\text{-}1 \cup \mathcal{NG}\text{-}2)$ and let its ABC-partition be $V(G) = A \cup B \cup C$. If $G \in \mathcal{NG}$-1, let $K = A \cup B$ and $S = C$ and otherwise, $G \in \mathcal{NG}$-2, and we let $K = B$ and $S = A \cup C$. In either case, using Theorem 2, we get a partition of $V(G)$ into a clique K and a stable set S, so G is a split graph. □

Not all split graphs are NG-graphs, for example, the path on 4 vertices is a split graph that is not an NG-graph. Indeed, the relationship between these classes, as well as the results in Remark 4 and Proposition 5, are shown in Fig. 2. We will define the classes of balanced and unbalanced split graphs in the next section.

Building on the work of Blázsik et al. [3], Maffray and Preissman [13] present a degree sequence characterization for NG-3 graphs. They combine this with the similar characterization for split graphs to get a linear-time recognition algorithm for pseudo-split graphs. We will discuss similar algorithms for NG-graphs in Section 3.

Finally, we note that Theorem 2 is very much related to the notion of graph decomposition that was studied systematically by Tyshkevich [17]. A graph is said to be *decomposable* if its vertex set can be partitioned into three parts A, B and C so that $A \neq \emptyset$ and $B \cup C \neq \emptyset$ and conditions (ii)–(v) of Theorem 2 are satisfied. Thus, every NG-graph is decomposable except when it is a single vertex or a 5-cycle. Chvátal and Hammer [5], Blázik et al. [3] and Barrus [1,2] also characterized various graph classes in terms of their decompositions.

Our main objective is to explore the connections between NG-graphs and split graphs. In Section 2, we study split graphs through the lens of NG-graphs. In particular, we determine which split graphs are NG-graphs and show how their