



Bivariate affine Gončarov polynomials

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ABSTRACT

Bivariate Gončarov polynomials are a basis of the solutions of the bivariate Gončarov Interpolation Problem in numerical analysis. A sequence of bivariate Gončarov polynomials is determined by a set of nodes $Z = \{(x_{i,j}, y_{i,j}) \in \mathbb{R}^2\}$ and is an affine sequence if Z is an affine transformation of the lattice grid $\mathbb{N}^2$, i.e., $(x_{i,j}, y_{i,j})^T = A(i, j)^T + (c_1, c_2)^T$ for some 2×2 matrix A and constants c_1, c_2 . In this paper we prove that a sequence of bivariate Gončarov polynomials is of binomial type if and only if it is an affine sequence with $c_1 = c_2 = 0$. Such polynomials form a higher-dimensional analog of the Abel polynomial $A_n(x; a) = x(x - an)^{n-1}$. We present explicit formulas for a general sequence of bivariate affine Gončarov polynomials and its exponential generating function, and use the algebraic properties of Gončarov polynomials to give some new two-dimensional generalizations of Abel identities.

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1. Introduction

Gončarov Interpolation is a problem in numerical analysis which asks for a polynomial of degree n whose i th derivative at point a_i equals a prefixed value b_i , for $i = 0, 1, \dots, n$. The sequence of Gončarov polynomials forms a basis of the solutions of Gončarov Interpolation. Explicitly, given a sequence of real numbers $a_0, a_1, \dots, a_n$, the Gončarov polynomial $g_n(x; a_0, \dots, a_{n-1})$ is defined by the biorthogonality relation:

$$\varepsilon(a_i) D^j g_n(x; a_0, a_1, \dots, a_{n-1}) = n! \delta_{i,n}$$

where D is the differential operator, and $\varepsilon(a)$ is evaluation at a . A special case of this is *Abel interpolation*, where the point $a_i = ai$ for a fixed constant a . In this case, the Gončarov polynomial $g_n(x; 0, a, 2a, \dots, (n-1)a)$ is called the *Abel polynomial* and denoted by $A_n(x; a)$, which has the explicit formula $A_n(x; a) = x(x - an)^{n-1}$. Abel polynomials are named after the Norwegian mathematician Niels Henrik Abel, and are closely related to counting of labeled trees and parking functions, (e.g. [1,12,22]).

A basic property for a sequence of Abel polynomials is that it is of binomial type, where a sequence $p_0(x), p_1(x), \dots$ of polynomials is of binomial type if and only if

$$p_m(x + y) = \sum_{i=0}^m \binom{m}{i} p_i(x) p_{m-i}(y).$$

The theory of polynomials of binomial type plays a fundamental role in *umbral calculus*, or *finite operator calculus*, an area of algebraic combinatorics pioneered by Rota and Mullin [13] and further developed by Rota and many others, for example, in

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[2,3,5,14,19–21], to list a few. Umbral calculus studies combinatorial problems by means of linear functionals on spaces of polynomials, and exhibits fascinating relationships between the formal power series methods of combinatorics, the calculus of finite differences, and the theory of special polynomials and identities. Polynomials of binomial type also occur naturally in the enumeration of binomial posets, and are related to graph colorings and acyclic orientations [23–25].

For multivariate polynomials, there is an analogous notion of polynomial sequences of binomial type, for which the explicit definition is given in Section 2. The basic theory of polynomial sequences of binomial type in several variables and their roles in multivariate umbral calculus was developed by Parrish [15]. Garsia and Joni [6,9] showed that this theory offers a natural setting for the study of compositional inverses of formal power series, and used higher dimensional versions of the Lagrange inversion theorem to derive certain new Steffensen-type formulas. Multivariate umbral calculus and its applications to linear recurrences are studied by Niederhausen [14].

The multivariate Gončarov polynomials were investigated by Khare, Lorentz and Yan in [11] as a basis of solutions for the multivariate Gončarov Interpolation. In the current paper we are concerned about the bivariate case unless otherwise stated. Bivariate Gončarov polynomials are related to pairs of integer sequences whose order statistics are bounded by certain weights along lattice paths in $\mathbb{N}^2$. To state the definition, let Z be a set of nodes in $\mathbb{R}^2$, that is, $Z = \{z_{i,j} = (x_{i,j}, y_{i,j}) \in \mathbb{R}^2 \mid 0 \leq i, 0 \leq j\}$, and let

$$\Pi_{m,n}^2 = \left\{ \sum_{i=0}^m \sum_{j=0}^n c_{i,j} x^i y^j \mid c_{i,j} \in \mathbb{R} \right\}$$

be the space of bivariate polynomials of coordinate degree (m, n) . The bivariate Gončarov polynomial $g_{m,n}((x, y); Z)$ is the unique polynomial in $\Pi_{m,n}^2$ satisfying

$$\frac{\partial^{i+j}}{\partial x^i \partial y^j} g_{m,n}(z_{i,j}) = m!n! \delta_{i,m} \delta_{j,n} \quad (1)$$

for all $0 \leq i \leq m$ and $0 \leq j \leq n$. Clearly $g_{m,n}((x, y); Z)$ depends only on the nodes $Z_{m,n} = \{z_{i,j} \in Z \mid 0 \leq i \leq m, 0 \leq j \leq n\}$. Many algebraic and combinatorial properties of $g_{m,n}((x, y); Z)$ are given in [11]. In particular, $g_{m,n}((x, y); Z)$ can be characterized by the following linear recursion

$$x^m y^n = \sum_{i=0}^m \sum_{j=0}^n \binom{m}{i} \binom{n}{j} x_{i,j}^{m-i} y_{i,j}^{n-j} g_{i,j}((x, y); Z), \quad (2)$$

and the Appell relation

$$e^{sx+ty} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} g_{m,n}((x, y); Z) \frac{s^m e^{x_{m,n}s}}{m!} \frac{t^n e^{y_{m,n}t}}{n!}. \quad (3)$$

Our first objective is to characterize bivariate Gončarov polynomials of binomial type, which would give a 2-dimensional generalization of the classical Abel polynomials. In Section 2 we present a necessary and sufficient condition: a sequence of bivariate Gončarov polynomials is of binomial type if and only if the set of nodes Z is a linear transformation of the standard lattice grid $\mathbb{N}^2$. We call such Gončarov polynomials *linear Gončarov polynomials*. In general, if Z is an affine transformation of $\mathbb{N}^2$, the corresponding Gončarov polynomials are called *affine Gončarov polynomials*. In Section 3 we calculate the compositional inverses of the functions appearing in the Appell relations of linear Gončarov polynomials. This allows us to transform the Appell relation into an exponential generating function for a sequence of linear/affine Gončarov polynomials. Section 4 contains explicit formulas for affine Gončarov polynomials. In the bivariate case the formula is complete. We also present some examples in the trivariate case. In the last section, we use the algebraic equations of Gončarov polynomials to derive various combinatorial identities in double summations, which offer a new family of 2-dimensional generalizations of Riordan's Abel identities [18].

We remark that in the literature, there are various generalizations of Abel polynomials, starting from Hurwitz's multinomial extensions [7] in 1902. The Abel–Hurwitz identities are further investigated by many researchers, including Riordan [18], Françon [4], Pitman [16,17], Kelmans and Postnikov [10], and are related to random mappings, subsets, forests, and forest volumes. There were also precedents of Abel polynomials deriving from finite differences. See [8] for a historical overview. Our generalization as linear Gončarov polynomials is a totally different approach which provides a new perspective to the subject. Our results are complementary to the existing research on Abel polynomials and reveal connections between combinatorics, polynomial systems, and interpolation theory.

2. Gončarov polynomials of binomial type

We adopt Parrish's definition of multivariate polynomials of binomial type [15].

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