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Local and global colorability of graphs

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ABSTRACT

It is shown that for any fixed $c \geq 3$ and r, the maximum possible chromatic number of a graph on n vertices in which every subgraph of radius at most r is c-colorable is $\tilde{\Theta}\left(n^{\frac{1}{r+1}}\right)$: it is $O\left((n/\log n)^{\frac{1}{r+1}}\right)$ and $\Omega\left(n^{\frac{1}{r+1}}/\log n\right)$. The proof is based on a careful analysis of the local and global colorability of random graphs and implies, in particular, that a random n-vertex graph with the right edge probability has typically a chromatic number as above and yet most balls of radius r in it are 2-degenerate.

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1. Introduction

1.1. Notation and definitions

For a simple undirected graph $G = (V, E)$ denote by $d(u, v)$ the *distance* between the vertices $u, v \in V$. The *degree* of a vertex $v \in V$, denoted by $\deg(v)$, is the number of its neighbors in G. A subset $V' \subseteq V$ is *independent* if no edge of G has both of its endpoints in V'. The *chromatic number* of G, denoted by $\chi(G)$, is the minimal number of independent subsets of V whose union covers V. A graph is *k-degenerate* if the minimum degree of every subgraph of it is at most k. In particular, a k-degenerate graph is $k+1$-colorable. We will work with random graphs $G_{n,p}$ in the Erdős–Rényi model, where there are n labeled vertices and each edge is included in the graph with probability p, independently of all other edges. We say that a property of G holds *with high probability (w.h.p.)* if this property holds with probability that tends to 1 as n tends to ∞. In this paper we are only interested in graphs with large chromatic number ℓ. It will be therefore equivalent to say that a property holds w.h.p. if its probability tends to 1 as ℓ tends to ∞.

Consider the following definition of r-local colorability:

Definition 1.1. Let r be a positive integer. Let $U_r(v, G)$ be the ball with radius r around $v \in V$ in G (i.e. the induced subgraph on all vertices in V whose distance from v is $\leq r$). Let

$$\ell\chi_r(G) = \max_{v \in V} \chi(U_r(v, G)) \tag{1.1}$$

denote the *r-local chromatic number* of G.

We also say that $U_r(v, G)$ is the r-ball around v in G. Finally, we define the main quantity discussed in this paper.

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Definition 1.2. For $\ell \geq c \geq 2$ and $r > 0$ let $f_c(\ell, r)$ be the greatest integer n such that every graph on n vertices whose r-local chromatic number is $\leq c$ is ℓ-colorable.

In other words, $f_c(\ell, r) + 1$ is the minimal number of vertices in a non-ℓ-colorable graph in which every r-ball is c-colorable. Note that $f_{c_1}(\ell, r) \leq f_{c_2}(\ell, r)$ for $c_1 \geq c_2$.

Definitions 1.1 and 1.2 appear explicitly in the paper of Bogdanov [5], but the quantity $f_c(\ell, r)$ itself has been investigated well before (see Sections 1.2 and 8 for more details).

The main goal of this paper is to estimate $f_c(\ell, r)$ for fixed c, r as ℓ tends to ∞. The main result is an upper bound tight up to a polylogarithmic factor for $f_c(\ell, r)$ for all fixed $c \geq 3$ and r.

1.2. Background and our contribution

Fix an $r > 0$. Somewhat surprisingly, the gap between $f_2(\ell, r)$ and $f_3(\ell, r)$ might be much bigger than the gap between $f_3(\ell, r)$ and $f_c(\ell, r)$ for any other fixed $c \geq 3$. Here is a short background on previous results regarding $f_c(\ell, r)$ for fixed c and r and large ℓ and our contributions to these problems.

Known upper bounds for $f_c(\ell, r)$ with fixed c, r, large ℓ

Erdős [7] showed that for sufficiently large m there exists a graph G with $m^{1+1/2k}$ vertices, that neither contains a cycle of length $\leq k$ nor an independent set of size m. As an easy consequence, G is not $m^{1/2k}$-colorable. Put $k = 2r + 1$, $\ell = m^{1/2k}$ and note that G has $n = m^{1+1/2k} = \ell^{2k+1} = \ell^{4r+3}$ vertices and $\ell\chi_r(G) \leq 2$ but is not ℓ-colorable. Hence

$$f_2(\ell, r) < \ell^{4r+3}.$$

A better estimate follows from the results of Krivelevich in [11]. Indeed, Theorem 1 in his paper implies that there exists an absolute positive constant c so that

$$f_2(\ell, r) < (c\ell \log \ell)^{2r}. \tag{1.2}$$

An upper bound for $f_3(\ell, r)$ can be derived from another result by Erdős [8]. Erdős worked with random graphs in the $G_{n,m}$ model, in which we consider random graphs with n vertices and exactly m edges. He showed that with probability > 0.8 and for $k \leq O(n^{1/3})$ large enough, $G_{n,kn}$ is not $\frac{k}{\log k}$-colorable but every subgraph spanned by $O(nk^{-3})$ vertices is 3-colorable.

It is easy to show that with high probability every r-ball in $G_{n,kn}$ has $O(k)^r$ vertices (later we prove and apply a similar result for graphs in the $G_{n,p}$ model). Combining the above results and taking $k = 2\ell \log \ell$, $n = O(k)^{r+3} = O(\ell \log \ell)^{r+3}$, it follows that with positive probability the graph $G_{n,kn}$ is not ℓ-colorable but every r-ball (and in fact every subgraph on $O(nk^{-3}) = O(k)^r$ vertices) is 3-colorable. Hence there exists $\beta > 0$ such that:

$$f_c(\ell, r) \leq f_3(\ell, r) \leq (\beta\ell \log \ell)^{r+3} \tag{1.3}$$

for large ℓ, fixed $r \geq 3$ and for $c \geq 3$.

Lower bounds for $f_c(\ell, r)$ with fixed c, r, large ℓ

Bogdanov [5] showed that for all $r > 0$ and $\ell \geq c \geq 2$:

$$f_c(\ell, r) \geq \frac{(\ell/c + r/2)(\ell/c + r/2 + 1) \cdots (\ell/c + 3r/2)}{(r+1)^{r+1}} \geq \left(\frac{\ell/c + r/2}{r+1}\right)^{r+1}. \tag{1.4}$$

When c and r are fixed, (1.4) implies that $f_c(\ell, r) = \Omega(\ell^{r+1})$. In Section 7.2, we improve the lower bound in this domain by a logarithmic factor: it is shown that for fixed $c \geq 2$ and r, $f_c(\ell, r) = \Omega\left(\ell^{r+1} \log \ell\right)$.

A special case - $f_c(\ell, 1)$ for fixed c, large ℓ

It is not hard to prove that $f_2(\ell, 1) = \Theta(\ell^2 \log \ell)$, using the known fact that the Ramsey number $R(t, 3)$ is $\Theta(t^2/\log t)$ (see [1,10]). In Section 7 we extend this result to every fixed $c \geq 2$, showing that $f_c(\ell, 1) = \Theta(\ell^2 \log \ell)$ for fixed $c \geq 2$.

The main contribution

The main result in this paper is an improved upper bound for $f_3(\ell, r)$. We show that for fixed $r > 0$:

$$f_3(\ell, r) \leq (10\ell \log \ell)^{r+1}. \tag{1.5}$$

Fix r and $c \geq 3$. By the result above, together with our new lower bound, it follows that there exists a constant $\delta = \delta(r, c)$ such that

$$\delta\ell^{r+1} \log \ell \leq f_c(\ell, r) \leq f_3(\ell, r) \leq (10\ell \log \ell)^{r+1}. \tag{1.6}$$

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