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Note

Upper bound on cubicity in terms of boxicity for graphs of low chromatic number

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ABSTRACT

The *boxicity* (respectively *cubicity*) of a graph G is the least integer k such that G can be represented as an intersection graph of axis-parallel k-dimensional boxes (respectively k-dimensional unit cubes) and is denoted by $\text{box}(G)$ (respectively $\text{cub}(G)$). It was shown by Adiga and Chandran (2010) that for any graph G, $\text{cub}(G) \leq \text{box}(G)\lceil \log_2 \alpha(G) \rceil$, where $\alpha(G)$ is the maximum size of an independent set in G. In this note we show that $\text{cub}(G) \leq 2\lceil \log_2 \chi(G) \rceil \text{box}(G) + \chi(G)\lceil \log_2 \alpha(G) \rceil$, where $\chi(G)$ is the chromatic number of G. This result can provide a much better upper bound than that of Adiga and Chandran for graph classes with bounded chromatic number. For example, for bipartite graphs we obtain $\text{cub}(G) \leq 2(\text{box}(G) + \lceil \log_2 \alpha(G) \rceil)$.

Moreover, we show that for every positive integer k, there exist graphs with chromatic number k such that for every $\epsilon > 0$, the value given by our upper bound is at most $(1+\epsilon)$ times their cubicity. Thus, our upper bound is almost tight.

1. Introduction

A graph G is an *intersection graph* of sets from a family of sets $\mathcal{F}$, if there exists $f : V(G) \to \mathcal{F}$ such that $uv \in E(G) \Leftrightarrow f(u) \cap f(v) \neq \emptyset$. An *interval graph* is an intersection graph in which the set assigned to each vertex is a closed interval on the real line. In other words, an interval graph is an intersection graph of closed intervals on the real line. Similarly, a *unit interval graph* is an intersection graph of closed unit intervals on the real line. An *axis-parallel k-dimensional box*, abbreviated to *k-box*, is a Cartesian product of the form $R_1 \times \cdots \times R_k$, where each R_i is an interval of the form $[a_i, b_i]$ on the real line. A *k-cube* is a k-box such that each R_i is an interval of the form $[a_i, a_i + 1]$. Given a graph G that is an intersection graph of k-boxes (respectively k-cubes), we call a function f a *k-box representation* (respectively *k-cube representation*) of G if f is a function that maps the vertices of G to k-boxes (respectively k-cubes) such that for any two vertices $u, v \in V(G)$, it holds that $uv \in E(G)$ if and only if $f(u) \cap f(v) \neq \emptyset$. The *boxicity* (respectively *cubicity*) of a graph G, denoted by $\text{box}(G)$ (respectively $\text{cub}(G)$), is the minimum non-negative integer k such that G has a k-box representation (respectively k-cube representation). Only complete graphs have boxicity (cubicity) 0. The class of graphs with boxicity at most 1 is the class of interval graphs, and the class of graphs with cubicity at most 1 is the class of unit interval graphs. When H_1 and H_2 are graphs such that $V(H_1) = V(H_2) = V(G)$ and $E(G) = E(H_1) \cap E(H_2)$, we write $G = H_1 \cap H_2$. The following observation was made by F.S. Roberts [7].

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Lemma 1 ([7]). *The boxicity of a non-complete graph G equals the least k such that there exist interval graphs $I_1, \dots, I_k$ such that $G = I_1 \cap \cdots \cap I_k$. The cubicity of a non-complete graph G equals the least k such that there exist unit interval graphs $U_1, \dots, U_k$ such that $G = U_1 \cap \cdots \cap U_k$.*

In the rest of the paper, we shall use n to denote the number of vertices of the graph being discussed. Logarithms will be to the base 2, unless otherwise specified.

Clearly, for any graph G, $\mathrm{box}(G) \le \mathrm{cub}(G)$. This suggests the following question: does there exist a function g such that $\mathrm{cub}(G) \le g(\mathrm{box}(G))$? It is easy to see that the answer is negative: consider a star on $n + 1$ vertices. Its cubicity is $\lceil \log n \rceil$ [7] whereas its boxicity is 1, since a star is an interval graph. Chandran and Mathew [5] showed that for any graph G, $\mathrm{cub}(G) \le \mathrm{box}(G) \lceil \log n \rceil$, where n is the number of vertices. Adiga and Chandran [3] improved this result by showing that n can be replaced by the size $\alpha(G)$ of a maximum independent set in G.

Lemma 2 ([3]). *For any graph G,* $\mathrm{cub}(G) \le \mathrm{box}(G) \lceil \log \alpha(G) \rceil$.

Remark. We demonstrate next that the bound in the above lemma is tight, i.e., given any two integers b and α, where $b \ge 1$ and $\alpha \ge 2$, we show that there exists a graph G with $\mathrm{box}(G) = b$, $\alpha(G) = \alpha$ and $\mathrm{cub}(G) = b\lceil \log \alpha \rceil$. It was shown by Roberts in [7] that for $p, n_1, \dots, n_p \ge 2$, the complete p-partite graph with n_i vertices in the ith part has boxicity p and cubicity $\sum_{i=1}^{p} \lceil \log n_i \rceil$. Therefore, when $b \ge 2$ the complete b-partite graph with α vertices in each part will serve our purpose. If $b = 1$, then a star with $\alpha + 1$ vertices has the desired properties, as mentioned above (see [7]).

In a loose sense, the two factors in the upper bound on $\mathrm{cub}(G)$ given in Lemma 2, namely $\lceil \log \alpha(G) \rceil$ and $\mathrm{box}(G)$, individually can make the cubicity of a graph high. Clearly $\mathrm{box}(G)$ is a lower bound for $\mathrm{cub}(G)$ since cubes are specialized boxes. The other term $\lceil \log \alpha(G) \rceil$ can make $\mathrm{cub}(G)$ high due to a geometric reason, captured in the so-called 'volume argument', which we reproduce here (also see [6]): let $\mathrm{cub}(G) = k$ and let d be the diameter of G. By considering the extreme intervals when a cube representation in k dimensions is projected in a fixed dimension, the projection in each dimension is contained in an interval of length $d + 1$. Hence the volume of a representation is at most $(d + 1)^k$. Also the volume is at least $\alpha(G)$, since it contains that many disjoint cubes. Hence, $\mathrm{cub}(G) \ge \lceil \log_{d+1} \alpha(G) \rceil$.

Thus $M = \max(\mathrm{box}(G), \lceil \log_{d+1} \alpha(G) \rceil) \le \mathrm{cub}(G)$. It is natural to ask whether there exists a function g such that $\mathrm{cub}(G) \le g(M)$. The answer is no, since we can increase the diameter of a graph without bound without affecting its cubicity. For example, if G is the graph obtained by identifying one endpoint of a path on $2n + 1$ vertices with the leaf of a star on $n+1$ vertices, it is easy to check that $\mathrm{box}(G) = 1$, $\alpha(G) = 2n$, the diameter d of G equals $2n+2$ and hence $M = \max\{\mathrm{box}(G), \lceil \log \alpha(G)/\log(d + 1) \rceil\} = 1$, whereas $\mathrm{cub}(G) = \lceil \log n \rceil$, which is far higher. (We have noted that the cubicity of a star with $n + 1$ vertices is $\lceil \log n \rceil$ [7], and identifying the endpoint of a path with a leaf of the star does not increase its cubicity.)

In this paper we ask a simpler question: let $\bar{M} = \max(\mathrm{box}(G), \lceil \log \alpha(G) \rceil)$. Lemma 2 tells us that $\mathrm{cub}(G) \in O(\bar{M}^2)$, and the remark after the lemma indicates that we cannot have anything better in general (choosing $\alpha = 2^b$ there illustrates the point). Can we show that $\mathrm{cub}(G) \in O(\bar{M})$ for some restricted graph classes? In this paper we show that if we restrict ourselves to classes of graphs whose chromatic number is bounded above by a constant, then such a result can indeed be proved. Our main theorem is a general upper bound for cubicity in terms of boxicity, the independence number, and the chromatic number:

Theorem 3. *If G is a graph with chromatic number $\chi(G)$ and independence number $\alpha(G)$, then* $\mathrm{cub}(G) \le 2 \lceil \log \chi(G) \rceil \mathrm{box}(G) + \chi(G) \lceil \log \alpha(G) \rceil$.

The proof of Theorem 3 is in Section 2. For graphs of low chromatic number, this result can be in general far better than that of Adiga et al. [3]. The most interesting case is that of bipartite graphs:

Corollary 4. *For a bipartite graph G,* $\mathrm{cub}(G) \le 2(\mathrm{box}(G) + \lceil \log \alpha(G) \rceil)$.

Remark. The reader may wonder whether the chromatic number is an upper bound for the boxicity of a graph, in which case Theorem 3 cannot be an improvement over Lemma 2. In fact, most graphs with fixed chromatic number have larger boxicity. In [2] it is shown that almost all balanced bipartite graphs (on $2n$ vertices) have boxicity $\Omega(n)$. The proof can be modified to show that almost all bipartite graphs with n vertices on one side and m vertices on the other have boxicity $\Omega(\min(n, m))$. Using the ideas from [2], it can be proved without much difficulty that, for any fixed k, boxicity is much greater than k for almost all balanced k-partite graphs. It also follows from [2] and [4] that almost all graphs have boxicity much larger than their chromatic number.

1.1. Preliminaries

A graph G is a co-bipartite graph if its complement $\overline{G}$ is a bipartite graph. Thus G is a co-bipartite graph if and only if the vertex set $V(G)$ can be partitioned into two cliques A and B. It is clear that $\alpha(G) \le 2$ when G is co-bipartite. Lemma 2 yields the following lemma.