Discrete Mathematics 339 (2016) 447–456

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

The square chromatic number of the torus

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ARTICLE INFO

Article history:
Received 28 January 2015
Received in revised form 1 September 2015
Accepted 2 September 2015
Available online 8 October 2015

Keywords:
2-distance colouring
Cartesian product
Torus

ABSTRACT

The square of a graph G denoted by G^2, is the graph with the same vertex set as G and edges linking pairs of vertices at distance at most 2 in G. The chromatic number of the square of the Cartesian product of two cycles was previously determined for some cases. In this paper, we determine the precise value of $\chi((C_m \Box C_n)^2)$ for all the remaining cases. We show that for all ordered pairs (m, n) except for $(7, 11)$ we have $\chi((C_m \Box C_n)^2) = \lceil \frac{|V((C_m \Box C_n)^2)|}{\alpha((C_m \Box C_n)^2)} \rceil$, where $\alpha(G)$ denotes the independent number of G. This settles a conjecture of Sopena and Wu (2010). We also show that the smallest integer k such that $\chi((C_m \Box C_n)^2) \leq 6$ for every $m, n \geq k$ is 10. This answers a question of Shao and Vesel (2013).

1. Introduction

A k-distance colouring of a graph is a colouring of its vertices such that any two vertices at distance at most k receive distinct colours. By the kth power of a graph G denoted by G^k, we mean the graph with the same set of vertices in which two vertices are adjacent when their distance in G is at most k. For a given graph G, it is of interest to find $\chi_k(G)$, the minimum number of colours necessary to have a k-distance colouring of G. Note that $\chi_k(G) = \chi(G^k)$, where χ stands for the ordinary chromatic number. The k-distance colouring of a graph was defined by Florica Kramer and Horst Kramer in 1969 [5,6] and has been studied extensively throughout the literature, for a survey see [7]. The Cartesian product of two graphs G and H, denoted by $G \Box H$, is a graph with vertex set $V(G) \times V(H)$ where two vertices (u, u') and (v, v') are adjacent when either $u = v$ and u' is adjacent with v' in H, or $u' = v'$ and u is adjacent with v in G. For $u \in V(H)$ let $G_u = G \Box \{u\}$ and for $v \in V(G)$ let $H_v = \{v\} \Box H$. We call G_u the uth column and H_v the vth row of $G \Box H$. For convenience let $T_{m,n}$ stand for $C_m \Box C_n$, and let $\alpha(G)$ stand for the size of an independent set in G with maximum cardinality. It is clear that for every graph G

$$\chi(G) \geq \frac{|V(G)|}{\alpha(G)}. \tag{1}$$

Sopena and Wu [10] proposed the following conjectures.

Conjecture 1 (*[10]*). *If $m, n \geq 3$, then $\chi(T^2_{m,n}) = \lceil \frac{|V(T^2_{m,n})|}{\alpha(T^2_{m,n})} \rceil$.*

Conjecture 2 (*[10]*). *There exists some constant c such that if $m, n \geq c$, then $\chi(T^2_{m,n}) \leq 6$.*

B.M. Kim, et al. in [4] have worked on the conjecture above, while Shao and Vesel [9] discovered a colouring showing that Conjecture 2 holds for $c = 40$. More formally, they proved the following theorem.

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http://dx.doi.org/10.1016/j.disc.2015.09.003
0012-365X/

Theorem A (*[9]*). *If* $m, n \geq 40$, *then* $\chi(T^2_{m,n}) \leq 6$.

Also, Shao and Vesel [9] proposed the following question.

Question 1 (*[9]*). *What is the smallest c such that if* $m, n \geq c$, *then* $\chi(T^2_{m,n}) \leq 6$?

Jamison and Matthews [2], Mahmoodian and Mousavi [8], also Sopena and Wu [10] independently proved the following theorem.

Theorem B. *For all m and n,* $\chi(T^2_{m,n}) = 5$ *if and only if both m and n are multiple of* 5.

The exact values of $\chi(T^2_{m,n})$ are determined in some papers for infinitely many other cases of (m, n), see [3,8–10]. Here we find $\chi(T^2_{m,n})$ for all remaining cases which were not known. Also we show that Conjecture 1 holds for all m and n except when $(m, n) = (7, 11)$ and also we show that the optimal value of c in Question 1 is 10.

The number c in Question 1 is equal to 10.

2. General results

We state a theorem from [3] which is a generalization of Theorem B. They state this theorem with different mathematical language and notation. An important corollary of this theorem will be used to prove Conjecture 1. We give a proof for one side of this theorem which can be instrumental in the proof of its corollary.

Theorem C (*[3]*). *Let* $G = C_{n_1} \Box C_{n_2} \Box \cdots \Box C_{n_k}$. *Then* $\chi(G^2) = 2k+1$ *if and only if* $\frac{|V(G^2)|}{\alpha(G^2)} = 2k+1$.

Proof. ($\Longrightarrow$) Let $V(G^2) = \{(x_1, x_2, \dots, x_k) \mid 0 \leq x_i \leq n_i - 1\}$. Assume that A_0 is an arbitrary independent set of G^2. Let $e_{+i} = (0, \dots, 0, 1, 0, \dots, 0)$ and $e_{-i} = (0, \dots, 0, -1, 0, \dots, 0)$, where for each i, $1 \leq i \leq k$, the ith coordinate is equal to $+1$ or -1. For each $x \in A_0$ let $A_x = \{x\} \cup \{x + e_{\pm i} \mid 1 \leq i \leq k\}$, where addition is taken modulo n_i. Now let $\mathcal{A} = \cup_{x \in A_0} A_x$. $\mathcal{A}$ is a collection of pairwise disjoint sets each of size $2k+1$. Therefore, $(2k+1)\alpha(G^2) \leq |V(G^2)|$. So

$$\chi(G^2) \geq \frac{|V(G^2)|}{\alpha(G^2)} \geq 2k+1. \tag{2}$$

By the hypothesis $\chi(G^2) = 2k+1$, hence $\frac{|V(G^2)|}{\alpha(G^2)} = 2k+1$.

For ($\Longleftarrow$) see [3]. □

Corollary 1. *Let* $G = C_{n_1} \Box C_{n_2} \Box \cdots \Box C_{n_k}$, *if* $\chi(G^2) \leq 2k+2$ *then* $\lceil \frac{|V(G^2)|}{\alpha(G^2)} \rceil = \chi(G^2)$.

Proof. If $\chi(G^2) = 2k+1$ then the statement follows from Theorem C. If $\chi(G^2) = 2k+2$ then by (1) $2k+2 = \chi(G^2) \geq \frac{|V(G^2)|}{\alpha(G^2)}$ and by (2) and Theorem C, we have $\frac{|V(G^2)|}{\alpha(G^2)} > 2k+1$. □

Let x and y be two integers, and

$$S(x, y) = \{\alpha x + \beta y \mid \alpha,\ \beta \text{ are nonnegative integers}\}.$$

Sylvester has shown the following lemma.

Lemma A (*[1,11]*). *Let x and y be relatively prime integers greater than* 1. *Then* $n \in S(x, y)$ *for all* $n \geq (x-1)(y-1)$. □

By applying Sylvester's Lemma, one can observe that

$$S(5, 6) = \mathbb{N} \setminus \{1, 2, 3, 4, 7, 8, 9, 13, 14, 19\}.$$

Theorem 1. *If* $m, n \in S(5, 6)$, *then* $\chi(T^2_{m,n}) \leq 6$.

Proof. The following patterns are proper 6-colourings of $T^2_{5,5}$, $T^2_{5,6}$, $T^2_{6,5}$, and $T^2_{6,6}$, respectively.

$$A = \begin{array}{|ccccc|} \hline 1&3&5&2&4\\ 2&4&1&3&5\\ 3&5&6&4&1\\ 4&2&3&5&6\\ 5&6&4&1&3\\ \hline \end{array} \quad B = \begin{array}{|cccccc|} \hline 1&3&5&2&6&4\\ 2&4&6&3&1&5\\ 3&5&1&4&2&6\\ 4&2&3&6&5&1\\ 5&6&4&1&3&2\\ \hline \end{array} \quad C = \begin{array}{|ccccc|} \hline 1&3&5&2&4\\ 2&4&6&3&5\\ 3&5&1&4&6\\ 4&6&2&5&1\\ 5&1&3&6&2\\ 6&2&4&1&3\\ \hline \end{array} \quad D = \begin{array}{|cccccc|} \hline 1&3&5&2&6&4\\ 2&4&6&3&1&5\\ 3&5&1&4&2&6\\ 4&6&2&5&3&1\\ 5&1&3&6&4&2\\ 6&2&4&1&5&3\\ \hline \end{array}.$$

$T^2_{5,5}$ $\qquad$ $T^2_{5,6}$ $\qquad$ $T^2_{6,5}$ $\qquad$ $T^2_{6,6}$