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The Cameron–Liebler problem for sets

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ABSTRACT

Cameron–Liebler line classes and Cameron–Liebler k-classes in $\mathrm{PG}(2k+1, q)$ are currently receiving a lot of attention. Here, links with the Erdős–Ko–Rado results in finite projective spaces occurred. We introduce here in this article the similar problem on Cameron–Liebler classes of sets, and solve this problem completely, by making links to the classical Erdős–Ko–Rado result on sets. We also present a characterisation theorem for the Cameron–Liebler classes of sets.

1. Introduction

In [4], Cameron and Liebler investigated the orbits of the projective groups $\mathrm{PGL}(n+1, q)$. For this purpose they introduced line classes in the projective space $\mathrm{PG}(3, q)$ with a specific property, which afterwards were called *Cameron–Liebler line classes*. A Cameron–Liebler line class $\mathcal{L}$ with parameter x in $\mathrm{PG}(3, q)$ is a set of $x(q^2+q+1)$ lines in $\mathrm{PG}(3, q)$ such that any line $\ell \in \mathcal{L}$ meets precisely $x(q+1)+q^2-1$ lines of $\mathcal{L}$ in a point and such that any line $\ell \notin \mathcal{L}$ meets precisely $x(q+1)$ lines of $\mathcal{L}$ in a point.

Many equivalent characterisations are known, of which we present one. For an overview we refer to [7, Theorem 3.2]. A *line spread* of $\mathrm{PG}(3, q)$ is a set of lines that form a partition of the point set of $\mathrm{PG}(3, q)$, i.e. each point of $\mathrm{PG}(3, q)$ is contained in precisely one line of the line spread. The lines of a line spread are necessarily pairwise skew. Now a line set $\mathcal{L}$ in $\mathrm{PG}(3, q)$ is a Cameron–Liebler line class with parameter x if and only if it has x lines in common with every line spread of $\mathrm{PG}(3, q)$.

The central problem for Cameron–Liebler line classes in $\mathrm{PG}(3, q)$, is to determine for which parameters x a Cameron–Liebler line class exists, and to classify the examples admitting a given parameter x. Constructions of Cameron–Liebler line classes and characterisation results were obtained in [3–5,9,13,16]. Recently several results were obtained through a new counting technique, see [11,12,14]. A complete classification is however not in sight.

Also recently, Cameron–Liebler k-classes in $\mathrm{PG}(2k+1, q)$ were introduced in [17] and Cameron–Liebler line classes in $\mathrm{PG}(n, q)$ were introduced in [12]. Both generalise the classical Cameron–Liebler line classes in $\mathrm{PG}(3, q)$.

Before describing the central topic of this article, we recall the concept of a q-analogue. In general a q-analogue is a mathematical identity, problem, theorem, . . ., that depends on a variable q and that generalises a known identity, problem, theorem, . . ., to which it reduces in the (right) limit $q \to 1$. In a combinatorial/geometrical setting it often arises by replacing a set and its subsets by a vector space and its subspaces. E.g. the q-binomial theorem is a q-analogue of the classical binomial theorem. In recent years there has been a lot of attention for q-analogues, see [1] amongst others.

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The Cameron–Liebler problem has not arisen as a q-analogue of a problem on sets, but has an interesting counterpart on sets that we will describe and investigate in this article. The definition builds on the spread definition of the classical Cameron–Liebler line classes and uses a classical set counterpart for spreads in a projective space. A subset of size k of a set will be called a *k-subset* or shortly a *k-set*.

Definition 1.1. A *k-uniform partition* of a finite set Ω, with $|\Omega| = n$ and $k \mid n$, is a set of pairwise disjoint k-subsets of Ω such that any element of Ω is contained in precisely one of the k-subsets.

Necessarily, a k-uniform partition of a finite set Ω, with $|\Omega| = n$, contains $\frac{n}{k}$ different k-subsets. This definition now allows us to present the definition of a Cameron–Liebler class of k-sets.

Definition 1.2. Let Ω be a finite set with $|\Omega| = n$ and let k be a divisor of n. A *Cameron–Liebler class of k-sets with parameter x* is a set of k-subsets of Ω which has x different k-subsets in common with every k-uniform partition of Ω.

Note that the q-analogue of the above definition is actually a Cameron–Liebler $(k-1)$-class in $\mathrm{PG}(n-1, q)$, a concept that has not been discussed before, but which is a straightforward generalisation of the Cameron–Liebler classes that have already been discussed.

We present two results on these Cameron–Liebler classes of subsets. In Theorem 2.5 we show that also for Cameron–Liebler classes of subsets many equivalent characterisations can be found. The second main theorem of this paper is the following classification result.

Theorem 1.3. *Let Ω be a finite set with $|\Omega| = n$ and let $\mathcal{L}$ be a Cameron–Liebler class of k-sets with parameter x in Ω, $k \geq 2$. If $n \geq 3k$ and $\mathcal{L}$ is nontrivial, then either $x = 1$ and $\mathcal{L}$ is the set of all k-subsets containing a fixed element or $x = \frac{n}{k} - 1$ and $\mathcal{L}$ is the set of all k-subsets not containing a fixed element.*

2. The classification result

The next result is the Erdős–Ko–Rado theorem, a classical result in combinatorics.

Theorem 2.1 ([8, Theorem 1] and [18]). *If $\mathcal{S}$ is a family of k-subsets in a set Ω with $|\Omega| = n$ and $n \geq 2k$, such that the elements of $\mathcal{S}$ are pairwise not disjoint, then $|\mathcal{S}| \leq \binom{n-1}{k-1}$. Moreover, if $n \geq 2k+1$, then equality holds if and only if $\mathcal{S}$ is the set of all k-subsets through a fixed element of Ω.*

Lemma 2.2. *Let Ω be a finite set with $|\Omega| = n$, and let $\mathcal{L}$ be a Cameron–Liebler class of k-sets with parameter x in Ω, with $k \mid n$.*

1. *The number of k-uniform partitions of Ω equals $\frac{n!}{\left(\frac{n}{k}\right)!(k!)^{\frac{n}{k}}}$.*
2. *The number of k-sets in $\mathcal{L}$ equals $x\binom{n-1}{k-1}$.*
3. *The set $\overline{\mathcal{L}}$ of k-subsets of Ω not belonging to $\mathcal{L}$ is a Cameron–Liebler class of k-sets with parameter $\frac{n}{k} - x$.*

Proof. 1. With every permutation (ordering) σ of the n elements of Ω, we can associate a partition P_σ in the following way: for every $i = 1, \ldots, \frac{n}{k}$ the elements on the positions $(i-1)k+1, (i-1)k+2 \ldots, ik$ form a k-subset of Ω, and these $\frac{n}{k}$ subsets are pairwise disjoint and form thus a k-uniform partition. Now, every partition can arise from $\left(\frac{n}{k}\right)!(k!)^{\frac{n}{k}}$ different permutations as the $\frac{n}{k}$ subsets can be permuted and each of these k-subsets can be permuted internally.

2. We perform a double counting of the tuples (C, P), with $C \in \mathcal{L}$, P a k-uniform partition and C a k-set in P. We find that

$$|\mathcal{L}|\frac{(n-k)!}{\left(\frac{n}{k}-1\right)!(k!)^{\frac{n-k}{k}}} = x\frac{n!}{\left(\frac{n}{k}\right)!(k!)^{\frac{n}{k}}} \quad \Rightarrow \quad |\mathcal{L}| = x\frac{n!k}{(n-k)!k!n} = x\binom{n-1}{k-1}.$$

3. Since every k-uniform partition of Ω contains x subsets belonging to $\mathcal{L}$, it contains $\frac{n}{k} - x$ subsets belonging to $\overline{\mathcal{L}}$. □

Example 2.3. Let Ω be a finite set with $|\Omega| = n$, and assume $k \mid n$. We give some examples of Cameron–Liebler classes of k-sets with parameter x. Note that $0 \leq x \leq \frac{n}{k}$.

- The empty set is obviously a Cameron–Liebler class of k-sets with parameter 0, and directly or via the last property in Lemma 2.2 it can be seen that the set of all k-subsets of Ω is a Cameron–Liebler class of k-sets with parameter $\frac{n}{k}$. These two examples are called the *trivial* Cameron–Liebler classes of k-sets.
- Let p be a given element of Ω. The set of k-subsets of Ω containing p is a Cameron–Liebler class of k-sets with parameter 1. Indeed, in every k-uniform partition of Ω there is exactly one k-subset containing p.
  Again using the last property of Lemma 2.2, we find that the set of all k-subsets of Ω not containing the element p is a Cameron–Liebler class of k-sets with parameter $\frac{n}{k} - 1$.

In the introduction we already mentioned that many equivalent characterisations for Cameron–Liebler classes in $\mathrm{PG}(3, q)$ are known. In Theorem 2.5 we show that this is also true for Cameron–Liebler classes of subsets. We did not mention the equivalent characterisations for the Cameron–Liebler sets in $\mathrm{PG}(3, q)$, but they arise as the q-analogues of the characterisations in Theorem 2.5.