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# Entropy of symmetric graphs

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ABSTRACT

Let $F_G(P)$ be a functional defined on the set of all the probability distributions on the vertex set of a graph $G$. We say that $G$ is *symmetric with respect to* $F_G(P)$ if the distribution $P^*$ maximizing $F_G(P)$ is uniform on $V(G)$. Using the combinatorial definition of the entropy of a graph in terms of its vertex packing polytope and the relationship between the graph entropy and fractional chromatic number, we prove that vertex-transitive graphs are symmetric with respect to graph entropy. As the main result of this paper, we prove that a perfect graph is symmetric with respect to graph entropy if and only if its vertices can be covered by disjoint copies of its maximum-size clique. Particularly, this means that a bipartite graph is symmetric with respect to graph entropy if and only if it has a perfect matching.



## 1. Introduction

The entropy of a graph is an information theoretic functional which is defined on a graph with a probability distribution on its vertex set. This functional was originally proposed by J. Körner in 1973 to study the minimum number of codewords required for representing an information source (see J. Körner [8]).

Let $VP(G)$ be the *vertex packing polytope* of a given graph $G$ which is the convex hull of the characteristic vectors of its independent sets. Let $|V(G)| = n$ and $P$ be a probability distribution on $V(G)$. Then the *entropy of* $G$ *with respect to the probability distribution* $P$ is defined as

$$H(G,P) = \min_{\mathbf{a}\in VP(G)} \sum_{i=1}^{n} p_i \log(1/a_i).$$

G. Simonyi [18] showed that the maximum of the graph entropy of a given graph over the probability distribution of its vertex set is equal to the logarithm of its fractional chromatic number. We say a graph is *symmetric with respect to graph entropy* if the uniform distribution maximizes its entropy. It is worth noting that the notion of a symmetric graph with respect to a functional was already defined by G. Greco in [7].

In this paper, we study some classes of graphs which are symmetric with respect to graph entropy. For example we show that vertex-transitive graphs are symmetric; however, our main results are the following theorems and corollary.

**Theorem.** *Let $G = (V, E)$ be a perfect graph and $P$ be a probability distribution on $V(G)$. Then $G$ is symmetric with respect to graph entropy $H(G, P)$ if and only if $G$ can be covered by disjoint copies of its maximum-size cliques.*

As a corollary to above theorem, we have

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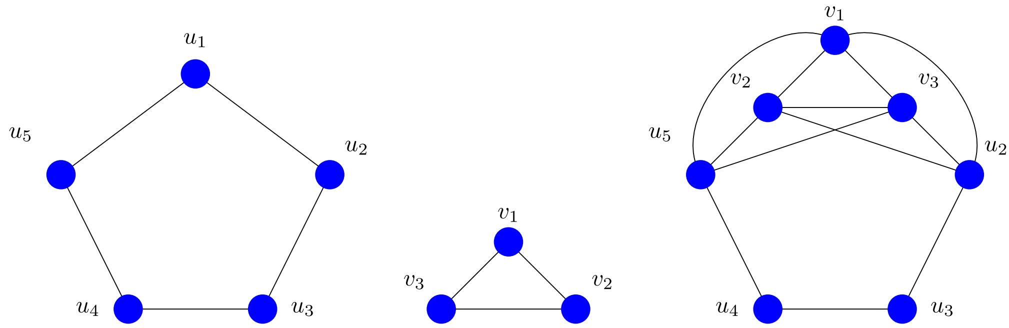

(a) A 5-cycle $G$. (b) A triangle $F$. (c) The graph $G_{u_1 \longleftarrow F}$.

**Fig. 1.**

**Corollary.** *Let $G$ be a bipartite graph with parts $A$ and $B$, and no isolated vertices. Then, uniform probability distribution $U$ over the vertices of $G$ maximizes $H(G, P)$ if and only if $G$ has a perfect matching.*

A. Schrijver [17] calls a graph $G$ a *k-graph* if it is $k$-regular and its fractional-edge chromatic number $\chi'_f(G)$ is equal to $k$. We show that.

**Theorem.** *Let $G$ be a $k$-graph with $k \geq 3$. Then the line graph of $G$ is symmetric with respect to graph entropy.*

As a corollary to this result we show that the line graph of every bridgeless cubic graph is symmetric with respect to graph entropy.

J. Körner investigated the basic properties of the graph entropy in several papers from 1973 till 1992 (see J. Körner [8–10,13,11,12]).

Let $F$ and $G$ be two graphs on the same vertex set $V$. Then the union of graphs $F$ and $G$ is the graph $F \cup G$ with vertex set $V$ and its edge set is the union of the edge set of graph $F$ and the edge set of graph $G$. That is

$$V(F \cup G) = V,$$
$$E(F \cup G) = E(F) \cup E(G).$$

The most important property of the entropy of a graph is that it is sub-additive with respect to the union of graphs, that is

$$H(F \cup G, P) \leq H(F, P) + H(G, P).$$

This leads to the application of graph entropy for graph covering problems.

The graph covering problem can be described as follows. Given a graph $G$ and a family of graphs $\mathcal{G}$ where each graph $G_i \in \mathcal{G}$ has the same vertex set as $G$, we want to cover the edge set of $G$ with the minimum number of graphs from $\mathcal{G}$. Using the sub-additivity of graph entropy one can obtain lower bounds on this number.

Graph entropy was implicitly used to give a non-trivial estimation in an unsolved combinatorial problem (see Fredman and Komlós [4]). It is worth mentioning that the notion of graph entropy was explicitly defined in J. Körner [9].

## 2. Preliminaries

### 2.1. Entropy of graphs

Let $G$ be a graph on vertex set $V(G) = \{1, \ldots, n\}$, and $P = (p_1, \ldots, p_n)$ be a probability distribution on $V(G)$.

**Remark 2.1.** Note that the function $\sum_{i=1}^{n} p_i \log \frac{1}{a_i}$ in the definition of the graph entropy is a convex function. It tends to infinity at the boundary of the non-negative orthant and tends monotonically to $-\infty$ along the rays from the origin. □

The main properties of graph entropy are *monotonicity*, *sub-additivity*, and *additivity under vertex substitution*. Monotonicity is formulated in the following lemma (see G. Simonyi [18]).

**Lemma 2.1** (*J. Körner*)**.** *Let $F$ be a spanning subgraph of a graph $G$. Then for any probability distribution $P$ we have $H(F, P) \leq H(G, P)$.*

We already explained the sub-additivity in the introduction section. It is worth noting that the sub-additivity was first recognized by Körner in [9].

The notion of substitution is defined as follows. Let $F$ and $G$ be two vertex disjoint graphs and $v$ be a vertex of $G$. We substitute $F$ for $v$ by deleting $v$ and joining all vertices of $F$ to those vertices of $G$ which have been adjacent with $v$. Let $G_{v \leftarrow F}$ be the resulting graph. The substitution operation is illustrated in Fig. 1.