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Note

Catalan-like numbers and Stieltjes moment sequences☆

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ABSTRACT

We provide sufficient conditions under which the Catalan-like numbers are Stieltjes moment sequences. As applications, we show that many well-known counting coefficients, including the Bell numbers, the Catalan numbers, the central binomial coefficients, the central Delannoy numbers, the factorial numbers, the large and little Schröder numbers, are Stieltjes moment sequences in a unified approach.

1. Introduction

A sequence $(m_n)_{n\geq 0}$ of numbers is said to be a *Stieltjes moment sequence* if it has the form

$$m_n = \int_0^{+\infty} x^n d\mu(x), \tag{1}$$

where μ is a non-negative measure on $[0, +\infty)$. It is well known that $(m_n)_{n\geq 0}$ is a Stieltjes moment sequence if and only if both $\det[m_{i+j}]_{0\leq i,j\leq n} \geq 0$ and $\det[m_{i+j+1}]_{0\leq i,j\leq n} \geq 0$ for all $n \geq 0$ [9, Theorem 1.3]. Another characterization for Stieltjes moment sequences comes from the theory of total positivity.

Let $A = [a_{n,k}]_{n,k\geq 0}$ be a finite or an infinite matrix. It is *totally positive* (*TP* for short), if its minors of all orders are nonnegative. Let $\alpha = (a_n)_{n\geq 0}$ be an infinite sequence of nonnegative numbers. Define the *Hankel matrix* $H(\alpha)$ of the sequence α by

$$H(\alpha) = [a_{i+j}]_{i,j\geq 0} = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & \cdots \\ a_1 & a_2 & a_3 & a_4 & \cdots \\ a_2 & a_3 & a_4 & a_5 & \cdots \\ a_3 & a_4 & a_5 & a_6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Then α is a Stieltjes moment sequence if and only if $H(\alpha)$ is totally positive (see [7, Theorem 4.4] for instance).

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Many counting coefficients are Stieltjes moment sequences. For example, the factorial numbers $n!$ form a Stieltjes moment sequence since

$$n! = \int_0^\infty x^n e^{-x} dx. \tag{2}$$

The Bell numbers B_n form a Stieltjes moment sequence since B_n can be interpreted as the nth moment of a Poisson distribution with expected value 1 by Dobinski's formula

$$B_n = \frac{1}{e} \sum_{k \geq 0} \frac{k^n}{k!}.$$

The Catalan numbers $C_n = \binom{2n}{n}/(n+1)$ form a Stieltjes moment sequence since

$$\det[C_{i+j}]_{0 \leq i,j \leq n} = \det[C_{i+j+1}]_{0 \leq i,j \leq n} = 1, \quad n = 0, 1, 2, \ldots$$

(see Aigner [1] for instance). Bennett [3] showed that the central Delannoy numbers D_n and the little Schröder numbers S_n form Stieltjes moment sequences by means of their generating functions (see Remark 2.11 and Example 2.12). All these counting coefficients are the so-called Catalan-like numbers. In this note we provide sufficient conditions such that the Catalan-like numbers are Stieltjes moment sequences by the total positivity of the associated Hankel matrices. As applications, we show that the Bell numbers, the Catalan numbers, the central binomial coefficients, the central Delannoy numbers, the factorial numbers, the large and little Schröder numbers are Stieltjes moment sequences in a unified approach.

2. Main results and applications

Let $\sigma = (s_k)_{k \geq 0}$ and $\tau = (t_k)_{k \geq 1}$ be two sequences of nonnegative numbers and define an infinite lower triangular matrix $R := R^{\sigma,\tau} = [r_{n,k}]_{n,k \geq 0}$ by the recurrence

$$r_{0,0} = 1, \qquad r_{n+1,k} = r_{n,k-1} + s_k r_{n,k} + t_{k+1} r_{n,k+1}, \tag{3}$$

where $r_{n,k} = 0$ unless $n \geq k \geq 0$. Following Aigner [2], we say that $R^{\sigma,\tau}$ is the *recursive matrix* and $r_{n,0}$ are the *Catalan-like numbers* corresponding to (σ, τ).

The Catalan-like numbers unify many well-known counting coefficients, such as

(1) the Catalan numbers C_n if $\sigma = (1, 2, 2, \ldots)$ and $\tau = (1, 1, 1, \ldots)$;
(2) the central binomial coefficients $\binom{2n}{n}$ if $\sigma = (2, 2, 2, \ldots)$ and $\tau = (2, 1, 1, \ldots)$;
(3) the central Delannoy numbers D_n if $\sigma = (3, 3, 3, \ldots)$ and $\tau = (4, 2, 2, \ldots)$;
(4) the large Schröder numbers r_n if $\sigma = (2, 3, 3 \ldots)$ and $\tau = (2, 2, 2 \ldots)$;
(5) the little Schröder numbers S_n if $\sigma = (1, 3, 3 \ldots)$ and $\tau = (2, 2, 2 \ldots)$;
(6) the (restricted) hexagonal numbers h_n if $\sigma = (3, 3, 3 \ldots)$ and $\tau = (1, 1, 1, \ldots)$;
(7) the Bell numbers B_n if $\sigma = \tau = (1, 2, 3, 4, \ldots)$;
(8) the factorial numbers $n!$ if $\sigma = (1, 3, 5, 7, \ldots)$ and $\tau = (1, 4, 9, 16, \ldots)$.

Rewrite the recursive relation (3) as

$$\begin{bmatrix} r_{1,0} & r_{1,1} & & & \\ r_{2,0} & r_{2,1} & r_{2,2} & & \\ r_{3,0} & r_{3,1} & r_{3,2} & r_{3,3} & \\ & & \cdots & & \ddots \end{bmatrix} = \begin{bmatrix} r_{0,0} & & & \\ r_{1,0} & r_{1,1} & & \\ r_{2,0} & r_{2,1} & r_{2,2} & \\ & \cdots & & \ddots \end{bmatrix} \begin{bmatrix} s_0 & 1 & & \\ t_1 & s_1 & 1 & \\ & t_2 & s_2 & \ddots \\ & & \ddots & \ddots \end{bmatrix},$$

or briefly,

$$\overline{R} = RJ$$

where $\overline{R}$ is obtained from R by deleting the 0th row and J is the tridiagonal matrix

$$J := J^{\sigma,\tau} = \begin{bmatrix} s_0 & 1 & & & \\ t_1 & s_1 & 1 & & \\ & t_2 & s_2 & 1 & \\ & & t_3 & s_3 & \ddots \\ & & & \ddots & \ddots \end{bmatrix}.$$

Clearly, the recursive relation (3) is decided completely by the tridiagonal matrix J. Call J the *coefficient matrix* of the recursive relation (3).