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Note

Forbidden induced subgraphs for bounded p-intersection number

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ABSTRACT

A graph G has p-intersection number at most d if it is possible to assign to every vertex u of G, a subset $S(u)$ of some ground set U with $|U| = d$ in such a way that distinct vertices u and v of G are adjacent in G if and only if $|S(u) \cap S(v)| \geq p$. We show that every minimal forbidden induced subgraph for the hereditary class $\mathcal{G}(d, p)$ of graphs whose p-intersection number is at most d, has order at most $\left(2^d + 1\right)^2$, and that the exponential dependence on d in this upper bound is necessary. For $p \in \{d - 1, d - 2\}$, we provide more explicit results characterizing the graphs in $\mathcal{G}(d, p)$ without isolated/universal vertices using forbidden induced subgraphs.

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1. Introduction

Intersection representations of graphs are among the most important graph representations and lead to some famous and well studied graph classes [9]. As a generalization of intersection representations, Jacobson et al. [6] introduced p-intersection representations. For a positive integer p, a *p-intersection representation* of a graph G is a function $S : V(G) \to 2^U$ assigning to every vertex u of G, a subset $S(u)$ of some ground set U in such a way that distinct vertices u and v are adjacent in G if and only if $|S(u) \cap S(v)| \geq p$. The choice $p = 1$ leads to classical intersection representations of graphs. Since every graph has a p-intersection representation for every p, it makes sense to study the *p-intersection number* $\Theta_p(G)$ of G defined as the minimum cardinality of a set U for which G has a p-intersection representation $S : V(G) \to 2^U$ with ground set U. The 1-intersection number was first studied by Erdős et al. [3] who observed that $\Theta_1(G) \leq d$ if and only if there are d cliques in G such that every edge of G belongs to at least one of these cliques. Kou et al. [8] showed that, unless P = NP, there is no polynomial-time algorithm to approximate $\Theta_1(G)$ with ratio better than 2. Most of the research on $\Theta_p(G)$ focused on estimates for special graphs such as paths, trees, bounded degree graphs, complete bipartite graphs [1,2,5,9].

In the present paper we consider the classes

$$\mathcal{G}(d, p) = \{G : \Theta_p(G) \leq d\}$$

of graphs for positive integers d and p. Clearly, $\mathcal{G}(d, p)$ is a hereditary class of graph and can therefore be characterized by minimal forbidden induced subgraphs. We give an upper bound on the order of minimal forbidden induced subgraphs for

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$\mathcal{G}(d, p)$. In principle, for every choice of d and p, this leads to a finite procedure that determines the complete list of minimal forbidden induced subgraphs for $\mathcal{G}(d, p)$. Nevertheless, unless d and p are rather restricted, this procedure is impractical. For $p \in \{d-1, d-2\}$, we provide more explicit results.

Considering the incidence vectors of the involved subsets of the ground set, it is easy to see that some graph G has a p-intersection representation $S : V(G) \to 2^U$ with $d = |U|$ if and only if there is a function $f : V(G) \to \{0, 1\}^d$ such that distinct vertices u and v are adjacent in G exactly if the dot product $f(u) \cdot f(v)$ of $f(u)$ and $f(v)$ is at least p. We refer to such a function as a *binary dot product representation of dimension d with threshold p*. Clearly, $\Theta_p(G)$ is the minimum d such that G has a binary dot product representation of dimension d with threshold p. Dot product representations using real vectors and thresholds were studied for instance in [4,7].

We only consider finite, simple, and undirected graphs, and use standard terminology and notation. The *vertex set* and the *edge set* of a graph G are denoted by $V(G)$ and $E(G)$, respectively. For a vertex u of a graph G, the *neighborhood* $N_G(u)$ of u in G is the set of vertices that are adjacent to u, that is, $N_G(u) = \{v \in V(G) : uv \in E(G)\}$. The *closed neighborhood* $N_G[u]$ of u in G is the set $N_G(u) \cup \{u\}$. A vertex without neighbors is *isolated* and a vertex whose closed neighborhood is the entire vertex set is *universal*. Two vertices u and v of a graph G are *twins* if $N_G[u] = N_G[v]$. A set of pairwise adjacent vertices is a *clique*, and a set of pairwise non-adjacent vertices is an *independent set*. The *complement* of a graph G is denoted by $\bar{G}$.

2. Results

Our first goal is to obtain an upper bound on the order of minimal forbidden induced subgraphs for $\mathcal{G}(d, p)$. In fact, we consider slightly more general classes of graphs.

For a graph G_0 and a partition $V(G_0) = C \cup I$ of its vertex set, let $\mathcal{G}(G_0; C, I)$ denote the class of graphs that arise from G by

- replacing every vertex u in C by a possibly empty clique C_u, and
- replacing every vertex u in I by a possibly empty independent set I_u.

Clearly, $\mathcal{G}(G_0; C, I)$ is a hereditary class of graphs. Note that replacing a vertex by a clique is also known as a *vertex expansion*, and replacing a vertex by an independent set is also known as a *vertex multiplication*.

If d and p are positive integers, $G_{(d,p)}$ is the graph of order 2^d for which the bijection $f : V\left(G_{(d,p)}\right) \to \{0, 1\}^d$ is a binary dot product representation of dimension d with threshold p,

$$C_{(d,p)} = \left\{\mathbf{x} \in \{0, 1\}^d : \mathbf{x} \cdot \mathbf{x} \geq p\right\}, \quad \text{and}$$
$$I_{(d,p)} = \{0, 1\}^d \setminus C_{(d,p)} = \left\{\mathbf{x} \in \{0, 1\}^d : \mathbf{x} \cdot \mathbf{x} < p\right\},$$

then a given graph G has a binary d-dot representation with threshold p if and only if G belongs to $\mathcal{G}\left(G_{(d,p)}; C_{(d,p)}, I_{(d,p)}\right)$. If for example $d = 3$ and $p = 2$, then $G_{(3,2)}$ is the disjoint union of a claw $K_{1,3}$ and four isolated vertices, the set $C_{(3,2)}$ contains the vertices of the claw, and the set $I_{(3,2)}$ contains the four isolated vertices, that is, all graphs that have a binary 3-dot representation with threshold 2 arise from $G_{(3,2)}$ by replacing the vertices of the claw by cliques, and the isolated vertices by independent sets.

We bound the order of minimal forbidden induced subgraphs for $\mathcal{G}(G_0; C, I)$.

Theorem 1. *Let G_0 be a graph and let $V(G_0) = C \cup I$ be a partition of its vertex set. If H is a minimal forbidden induced subgraph of $\mathcal{G}(G_0; C, I)$, then the order of H is at most $4|C||I| + 2|C| + 2|I| + 1$. Specifically, $\mathcal{G}(G_0; C, I) = Forb(\mathcal{F})$ for a finite set $\mathcal{F}$ of graphs.*

Proof. First, we assume that there are at least $|I|+2$ vertices $u_1, \ldots, u_k$ of H that are twins. Since $H-u_k$ belongs to $\mathcal{G}(G_0; C, I)$, replacing the vertices u in C by suitable cliques C_u, and replacing the vertices u in I by suitable independent sets I_u, results in $H - u_k$. Since for every vertex u in I, the set I_u contains at most one of the vertices $u_1, \ldots, u_{k-1}$, and since $k - 1 \geq |I| + 1$, there is some vertex v in C such that $u_i \in C_v$ for some $i \in [k-1]$. Replacing the vertices in $V(G)\setminus\{v\}$ as before, and replacing the vertex v by the clique $C_v \cup \{u_k\}$ results in H, a contradiction. This implies that for every vertex u of H, there are at most $|I|$ distinct further vertices of H that have the same closed neighborhood as u. Similarly, suitably exchanging the roles of C and I in the above argument, we obtain that for every vertex u of H, there are at most $|C|$ distinct further vertices of H that have the same (open) neighborhood as u.

Let u^* be a vertex of H. Let $H' = H - u^*$. If there are at least $2|I|+3$ vertices of H' that have the same closed neighborhood in H', then at least $\lceil(2|I| + 3)/2\rceil = |I| + 2$ of these vertices have the same closed neighborhood in H, a contradiction. Therefore, for every vertex u of H', there are at most $2|I| + 1$ distinct further vertices of H' that have the same closed neighborhood as u, and, similarly, for every vertex u of H', there are at most $2|C| + 1$ distinct further vertices of H' that have the same neighborhood as u. Since H' belongs to $\mathcal{G}(G_0; C, I)$, replacing the vertices u in C by suitable cliques C'_u, and replacing the vertices u in I by suitable independent sets I'_u, results in H'. Since for every vertex $u \in C$, all vertices in C'_u have the same closed neighborhood in H', we have $|C'_u| \leq 2|I| + 2$. Since for every vertex $u \in I$, all vertices in I'_u have the same neighborhood in H', we have $|I'_u| \leq 2|C| + 2$. Altogether, we obtain $n(H) = n(H') + 1 \leq |C|(2|I| + 2) + |I|(2|C| + 2) + 1 = 4|C||I| + 2|C| + 2|I| + 1$. □

As a corollary, we obtain the desired upper bound on the order of minimal forbidden induced subgraphs for $\mathcal{G}(d, p)$.

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