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A characterization of minimal non-Seymour graphs

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ABSTRACT

A connected undirected graph G is a Seymour graph if the maximum number of edge disjoint T-cuts is equal to the cardinality of a minimum T-join for every even subset T of $V(G)$. Ageev, Kostochka, and Szigeti characterized Seymour graphs in 1997. In this paper, we characterize minimal non-Seymour graphs. More precisely, we show that minimal non-Seymour graphs can be completely described by two infinite families of graphs, and we provide a procedure to construct them. Our characterization also generalizes a theorem of Lovász concerning minimal nonbipartite matching-covered graphs.

1. Introduction

We denote an undirected connected graph G by a pair $(V(G), E(G))$, where $V(G)$ and $E(G)$ represent the vertex and edge sets of G, respectively. Loops and multiple edges are permitted. Let G be a graph, and let $X \subseteq V(G)$. We use ∂X to denote the set of edges with one end in X and the other in $V(G)\backslash X$. A *cut* in G is any set of the form ∂X for some $X \subseteq V(G)$. The graphs $G[X]$ and $G[V(G)\backslash X]$ are the *sides* of the cut ∂X. If $|X| = 1$, then the cut ∂X is *trivial*.

For $F \subseteq E(G)$ and $v \in V(G)$, we denote by $d_F(v)$ the number of incidences of edges in F with v. Let T be an even subset of $V(G)$. A T-*cut* is a set of the form ∂X for some subset X of $V(G)$ such that $|X \cap T|$ is odd. A T-*join* is a subset F of $E(G)$ such that $T = \{v \in V(G) : d_F(v) \text{ is odd}\}$. Since G is connected and $|T|$ is even, a T-join necessarily exists. If $|T| = 2$, then a minimum T-join is a path of minimum length joining the vertices of T. (We identify paths and circuits with their edge sets.) If $T = V(G)$ and G has a perfect matching, then any perfect matching is a minimum T-join.

Let $\nu(G, T)$ denote the maximum number of disjoint T-cuts, and let $\tau(G, T)$ denote the minimum size of a T-join in G. It is easy to see that each T-join has at least one edge in common with each T-cut. Therefore,

$$\nu(G, T) \leq \tau(G, T). \quad (1)$$

In general, we do not have equality in (1): $G = K_4$ and $T = V(K_4)$ is a counterexample. However, Seymour showed that bipartite [13] and series–parallel [12] graphs satisfy (1) with equality for every even subset T of the vertex set of G.

Motivated by these results, we call a graph G a *Seymour* graph if (1) holds with equality for all even subsets $T \subseteq V(G)$. Ageev, Kostochka, and Szigeti [1] presented a characterization of Seymour graphs in 1997. Their result is stated in terms of conservative functions and implies Seymour's results.

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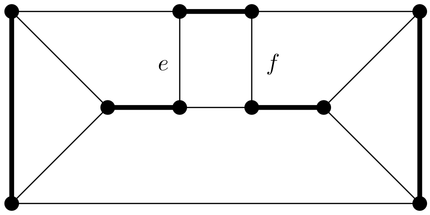

Fig. 1. A non-minimal non-Seymour graph, with negative edges bold.

A weight function $w : E(G) \to \{1, -1\}$ defined on the edges of G is *conservative* if the sum of the edge weights along any circuit is non-negative. For example, a matching in a graph defines a conservative function if we assign -1 to the edges of the matching and 1 to the remaining edges. A *conservatively-weighted graph* is an ordered pair (G, w) such that G is a nonempty connected graph with no isthmuses and w is a conservative weight function on $E(G)$. For convenience, we refer to (G, w) as a *conservative graph*. An edge e is *negative* if $w(e) = -1$; otherwise it is *positive*. Analogously, we define negative and positive sets of edges by considering the weight of a set of edges to be the sum of the weights of its members. The set of negative edges is denoted by $E^-(G)$. If $C \subseteq E(G)$ and $w(C) = 0$, then C is said to be *zero-weight*.

Let (G, w) be a conservative graph. If every edge of G lies in a zero-weight circuit then (G, w) is *join-covered*. Clearly, join-covered graphs have no loops.

Let (G, w) be a conservative graph. The *distance* (generated by w) between vertices x and y is given by the minimum weight (also called the w-minimum weight) of an x, y-path. We denote the distance between x and y by $d_w(x, y)$. In cases where there is no ambiguity we use the simpler notation $d(x, y)$. A *minimum-weight path* is a path whose weight is the minimum among all paths joining the same pair of vertices. Because weights may be negative, subpaths of minimum-weight paths are not necessarily minimum-weight.

Let C be a zero-weight circuit in a conservative graph (G, w). The weight function obtained from w by switching the signs on the edges of C is said to be obtained by *rotation* about C. In the case of a graph with a perfect matching M which is the set of negative edges, the zero-weight circuits are the alternating circuits with respect to M. Rotation about such an alternating circuit C results in a weighting where the negative edges are those that belong to the perfect matching $M \oplus C$ (the symmetric difference of M and C). The following lemma says that rotation about a zero-weight circuit results in a conservative graph with the same distance function.

Lemma 1.1 ([4]). *Let (G, w) be a conservative graph and C a zero-weight circuit of G. Let w' be the weight function obtained from w by rotation about C. Then w' is also conservative and $d_{w'}(x, y) = d_w(x, y)$, for all $x, y \in V(G)$.*

The result above naturally suggests the following definition: a weight function w' is *equivalent* to w if it is obtained from w by a sequence of rotations about zero-weight circuits. If w' is equivalent to w then, by Lemma 1.1, w' is conservative and $d_{w'}(x, y) = d_w(x, y)$, for all $x, y \in V(G)$.

A subgraph H of a join-covered graph (G, w) is *conformal* if there exists w' equivalent to w such that (H, w'_H) is a join-covered graph, where w'_H denotes the restriction of w' to $E(H)$. In the case of a matching-covered graph G, a conformal subgraph H is a matching-covered subgraph such that $G - V(H)$ has a perfect matching. The conformal property is transitive, that is, if H is a conformal subgraph of G and K is a conformal subgraph of H, then K is also a conformal subgraph of G.

A join-covered graph not induced by a circuit is *minimal* with respect to some specified property if any proper conformal subgraph with that property is induced by a circuit.

The following result is the characterization of Seymour graphs by Ageev et al.

Theorem 1.2 ([1]). *A graph G is non-Seymour if and only if for some conservative weight function there exist zero-weight circuits C_1 and C_2 such that the graph $G[C_1 \cup C_2]$ is nonbipartite.*

Not every graph satisfying the property stated in Theorem 1.2 is minimal. The graph of Fig. 1 is non-Seymour but it is not minimal, since removal of edges e and f results in a non-Seymour graph.

In this paper, we use Theorem 1.2 to characterize minimal non-Seymour graphs. More precisely, we show that minimal non-Seymour graphs can be completely described by two infinite families of graphs, and we provide a procedure to construct them. These two families are in fact minimal nonbipartite join-covered graphs. This fact is used to show that our characterization also generalizes a theorem of Lovász related to minimal nonbipartite matching-covered graphs.

In Section 2, we present the basic properties of conservative graphs. Section 3 is dedicated to the study of join-covered graphs and contains the most important tools used to prove the main result of this paper. Minimal bipartite join-covered graphs have been characterized in [3], but we present in Section 4 a slightly stronger version of that characterization which will be useful for the main result. In Section 5, we characterize minimal 2-connected nonbipartite join-covered graphs, a result that generalizes Lovász's characterization of minimal nonbipartite matching-covered graphs [5] (see also [7]). Finally, in Section 6 we use this result together with Theorem 1.2 to characterize minimal non-Seymour graphs.