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Intermingled ascending wave m-sets

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ABSTRACT

Given a coloring of $\mathbb{Z}^+$, we call a monochromatic set $A = \{a_1 < a_2 < \cdots < a_m\}$ an m-set. The diameter of A is $a_m - a_1$. Given two m-sets A and B, we say that they are non-overlapping if $\max(A) < \min(B)$ or $\max(B) < \min(A)$. The original study of non-overlapping m-sets, done by Bialostocki, Erdős, and Lefmann, concerned non-decreasing diameters. We investigate an "intermingling" of certain subset diameters of non-overlapping m-sets. In particular, we show that, for every integer $m \geq 2$, the minimum integer $n(m)$ such that every 2-coloring of $[1, n(m)]$ admits two m-sets $\{a_1 < a_2 < \cdots < a_m\}$ and $\{b_1 < b_2 < \cdots < b_m\}$ with $a_m < b_1$, such that $b_1 - a_1 \leq b_2 - a_2 \leq \cdots \leq b_m - a_m$ is $n(m) = 6m - 5$. The r-coloring case is also investigated.

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1. Introduction

This article falls under the general heading of Ramsey theory on the integers. In order to discuss some of the history, we first have need of some definitions and notation. Recall that a 2-coloring of a set S is a map $\chi : S \to T$, where $|T| = 2$. We will use $T = \{G, R\}$ (to stand for green and red).

Definition 1. Let $m \geq 2$ be an integer and consider an arbitrary 2-coloring χ of $\mathbb{Z}^+$. We say that $A \subseteq \mathbb{Z}^+$ is an *m-set* if $|A| = m$ and A is monochromatic under χ (i.e., $|\chi(A)| = 1$).

Definition 2. For $A = \{a_1 < a_2 < \cdots < a_m\}$ we call $a_m - a_1$ the *diameter of* A and write $\mathrm{diam}(A)$; we refer to the differences $a_{i+1} - a_i$, $1 \leq i \leq m - 1$, as *gaps*.

Definition 3. For A and B both m-sets (possibly of different colors, but both monochromatic by definition), we say that $A = \{a_1 < a_2 < \cdots < a_m\}$ and $B = \{b_1 < b_2 < \cdots < b_m\}$ are *non-overlapping* if either $a_m < b_1$ or $b_m < a_1$. When we have $a_m < b_1$ we write $A \prec B$.

The research into m-sets with diameter restrictions was started by Bialostocki, Erdős, and Lefmann. In their paper [3] they proved, in particular, that for $m \geq 2$, the integer $s = s(m) = 5m - 3$ is the minimum integer such that every 2-coloring of $[1, s]$ admits two m-sets A and B with $A \prec B$, such that $\mathrm{diam}(A) \leq \mathrm{diam}(B)$. They also provided the associated Ramsey-type number for three colors, while Bernstein, Grynkiewicz, and Yerger [2] provide a formula in the two color, three set situation, and Grynkiewicz [6] proved that $12m - 9$ is the correct interval length for two colors for four m-sets with non-decreasing

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diameters. Bollobás, Erdős, and Jin [5] investigated the situation for strictly increasing diameters. If we insist that two m-sets have equal diameters, the answer is still unknown; Bialostocki and Wilson [4] conjecture that the correct interval length is $6m - 4$. Schultz [8] investigated the minimal interval length for r-colorings that admit m-sets with $2 \cdot \mathrm{diam}(A) \leq \mathrm{diam}(B)$, finding exact values for $r = 2, 3, 4$.

Perhaps the closest diameter requirement to what we study in this article was done by Grynkiewicz and Sabar [7], in which they study pairs of non-overlapping m-sets that satisfy $b_j - a_j \geq b_1 - a_1$ for a fixed $j \in \{2, 3, \ldots, m\}$. Our requirement on the relationship between non-overlapping m-sets is examined in Section 2. There, not only do we require non-decreasing diameters, but also non-decreasing gaps between corresponding pairs of elements in each set. Specifically, for $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_m\}$ we require

$$b_1 - a_1 \leq b_2 - a_2 \leq \cdots \leq b_{m-1} - a_{m-1} \leq b_m - a_m. \qquad (*)$$

Note that by considering only the first and last arguments above we have $b_1 - a_1 \leq b_m - a_m$, which is equivalent to $\mathrm{diam}(A) \leq \mathrm{diam}(B)$. Also note that $(*)$ is equivalent to the conditions $b_j - b_{j-1} \geq a_j - a_{j-1}$ for $j = 2, 3, \ldots, m$. Hence, this is also a refinement of what was studied by Grynkiewicz and Sabar [7].

Based on our refinement, we make the following definition.

Definition 4. Let $r \geq 2$ be an integer. Define $n(m; r)$ to be the minimum integer n such that every r-coloring of $[1, n]$ admits two non-overlapping m-sets $\{a_1, a_2, \ldots, a_m\} \prec \{b_1, b_2, \ldots, b_m\}$ with $b_1 - a_1 \leq b_2 - a_2 \leq \cdots \leq b_m - a_m$.

The main result is in Section 2; namely, $n(m; 2) = 6m - 5$. In Section 3 we investigate $n(m; r)$ and end with some open questions.

Remark. We refer to sets that satisfy $(*)$ as *intermingled ascending wave m-sets.* A set $S = \{s_1, s_2, \ldots\}$ is called an *ascending wave* if $s_{j+1} - s_j \geq s_j - s_{j-1}$ for $j = 2, 3, \ldots$ (see [1]). In other words, successive gaps between adjacent elements of S are non-decreasing (which has been labeled in the literature as "ascending" instead of the more appropriate "non-descending"). We are intermingling the sets by considering $b_j - a_j$ and requiring that these "intermingled gaps" be non-decreasing.

Notation. Throughout the paper, unless otherwise stated, we will use $A = \{a_1 < a_2 < \cdots < a_m\}$ and $B = \{b_1 < b_2 < \cdots < b_m\}$, sometimes without explicitly stating so. Since we will be dealing strictly with the positive integers, we use the notation $[i, j] = \{i, i + 1, \ldots, j\}$ for positive integers $i < j$. For all of our 2-colorings, we use green and red for the colors and denote these colors by G and R, respectively.

2. Intermingled ascending wave *m*-sets using two colors

We prove that $n(m) = 6m - 5$ in the standard way: by providing matching lower and upper bounds for $n(m)$. We start with the lower bound.

Lemma 5. *For $m \geq 2$, we have $n(m) \geq 6m - 5$.*

Proof. Consider the following 2-coloring $\gamma : [1, 6m - 6] \to \{G, R\}$ defined by

$$\underbrace{RGRG\ldots GR}_{2m-1}\underbrace{GGRRGGRR\ldots GGR}_{4m-5}.$$

We will show that γ does not admit two m-sets $A \prec B$ that satisfy $(*)$.

Consider the case when $A \subseteq [1, 2m]$. Then we have $a_j - a_{j-1} = 2$ for all $j \in [2, m]$. Hence, we must have $B \subseteq [2m + 1, 6m - 6]$ with $b_j - b_{j-1} \geq 2$. Thus, B contains at most one element from each of the following $2m - 2$ monochromatic sets: $\{2m + 1\}, [2m + 2, 2m + 3], \ldots, [4m - 7, 4m - 6], \{4m - 5\}$. Each color appears in exactly $m - 1$ of these sets. Hence, B cannot have m elements.

Next, consider the case when $A \not\subseteq [1, 2m]$. In this situation, A may contain gaps of size 1. Let there be exactly j elements of A that are in $[2m, 6m - 6]$, where $j \geq 2$. This gives us $a_m \geq 2m + 2j - 3$ so that $b_1 \geq 2m + 2j - 2$. Turning our attention to B, we see that the $m - j$ gaps defined by the first $m - j + 1$ elements of B must have average gap size at least 4 since the corresponding gaps defined by the first $m - j + 1$ elements of A are necessarily greater than 1. Hence, we have $b_{m-j+1} \geq b_1 + 4(m - j) \geq (2m + 2j - 2) + 4(m - j) = 6m - 2j - 2$ as we can only use at most one element from any four consecutive elements with color pattern $GGRR$ or $RRGG$. There are at most $2j - 4$ elements larger than b_{m-j+1}, of which at most $j - 2$ are the same color as $b_1, b_2, \ldots, b_{m-j+1}$. However, we require $j - 1$ more elements in order for B to contain m total elements.

In both cases, under γ we do not have m-sets $A \prec B$ that satisfy $(*)$. Hence, we conclude that $n(m) > 6m - 6$. □

We now move on to our main result.

Theorem 6. *For $m \geq 2$, let $n = n(m)$ be the minimal integer such that every 2-coloring of $[1, n]$ admits m-sets $A \prec B$ that satisfy $b_{j+1} - a_{j+1} \geq b_j - a_j$ for $1 \leq j \leq m - 1$. Then $n(m) = 6m - 5$.*

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