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Ramsey numbers involving a long path

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ABSTRACT

Let H be a graph, and $\sigma(H)$ the minimum size of a color class among all proper vertex-coloring of H with $\chi(H)$ colors. A connected graph G with $|G| \geq \sigma(H)$ is said to be H-good if $R(G, H) = (\chi(H) - 1)(|G| - 1) + \sigma(H)$. In this note, we shall show that if $n \geq 8|H| + 3\sigma^2(H) + c\chi^8(H)$, then P_n is H-good, where $c = 10^{14}$.

1. Introduction

Let K_N be a complete graph of order N. For graphs G and H, the *Ramsey number* $R(G, H)$ is the minimum N such that every red–blue edge-coloring of K_N contains a red G or a blue H. There has been a lot of work put into studying $R(G, H)$ [8], but nevertheless it is only known in cases where at least one of the graphs is structurally simple, see [12].

As usual, we write $|H|$ the order of H, $\chi(H)$ the chromatic number of H, and $\sigma(H)$ the chromatic surplus, the minimum size of a color class among all proper vertex-colorings of H using $\chi(H)$ colors. Chvátal and Harary [6], and Burr [2] had the following lower bound.

Lemma 1. *Let G and H be graphs. If G is connected and $|G| \geq \sigma(H)$, then*

$$R(G, H) \geq (\chi(H) - 1)(|G| - 1) + \sigma(H).$$

A connected graph G is said to be *H-good* if the inequality in Lemma 1 becomes an equality, and a K_p-good graph is said to be p-good. Moreover, a family $\mathcal{F}$ of graphs is said to be p-good if all sufficiently large graphs in $\mathcal{F}$ are p-good. Chvátal [5] proved that a tree is p-good for any p and thus the family of trees is p-good.

Let P_n^k be the kth power of P_n, whose edges consist of pairs $\{x, y\}$ with distance in P_n at most k. The *bandwidth* of G, denoted by $bw(G)$, is defined to be the smallest integer k such that G is a subgraph of P_n^k, where $n = |G|$. A family of connected graphs is said to be always-good if for any graph H, any large graph in the family is H-good. Burr and Erdős [3] showed that for any k, the family of connected graphs G with $bw(G) \leq k$ is p-good for all p. Later, Allen, Brightwell and Skokan [1], and Nikiforov and Rousseau [10] showed much more than [3]. Both [1] and [10] are trying to explain which sparse graph families will be H-good (dense families never are for interesting H), the former takes "sparse" to mean bounded degree and gets reasonably close to the sharp characterization in terms of expansion (recently Choongbum Lee [9] sharpened this further), the latter works in the harder world of degenerate graphs, where H is necessarily restricted and the characterization is weaker.

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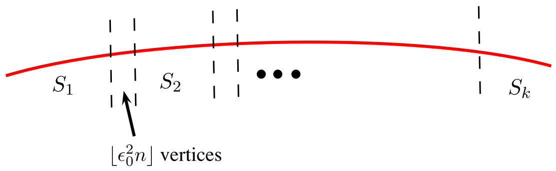

Fig. 1. Segments $S_1, \ldots, S_k$ in a path.

To an always-good family $\mathcal{F}$, a natural question is when H is fixed, how large a graph G in $\mathcal{F}$ should be to guarantee that G is H-good? Specifically, how large n should be such that P_n is H-good? For the latter, Allen, Brightwell and Skokan conjectured that $n \geq \chi(H)|H|$ is enough.

Conjecture 2 ([1]). *Let H be a graph. If $n \geq \chi(H)|H|$, then*

$$R(P_n, H) = (\chi(H) - 1)(n - 1) + \sigma(H).$$

Let $W_m = K_1 + C_m$. It is shown in [4] that P_n is W_m-good if m is odd and $n \geq m - 1 \geq 2$, or m is even and $n \geq m - 1 \geq 3$; Pokrovskiy [11] proved that P_n is P_n^k-good, providing evidences for Conjecture 2. In this note, we prove the following results.

Theorem 3. *Let H be a graph. If $n \geq 8|H| + 3\sigma^2(H) + c\chi^8(H)$, where $c = 10^{14}$, then*

$$R(P_n, H) = (\chi(H) - 1)(n - 1) + \sigma(H).$$

As a consequence of Theorem 3, one can certainly obtain that for any H, P_n is H-good for sufficiently large n, which was also obtained by Burr [2], and Erdős, Faudree, Rousseau, and Schelp [7]. Compared with $n \geq \Theta(|H|^2)$ in [2], Theorem 3 is meaningful only for bounded $\chi(H)$. If both $\chi(H)$ and $\sigma(H)$ are bounded, then it is $n \geq \Theta(|H|)$, a different type of parameter compared with that considered in Conjecture 2.

2. A Ramsey-theoretic result

To simplify the notation, when there is no danger of confusion, we will not distinguish between a subgraph and its vertex set.

As usual, denote by $K_\ell(m)$ the complete ℓ-partite graph with each part of size m. The following result is obtained by Pokrovskiy [11], in which the case $K_\ell(0)$ means its vertex set is empty.

Lemma 4. *For $\ell \geq 1$ and $N \geq \ell$, every red–blue edge-colored K_N can be partitioned into ℓ disjoint red-paths and a blue $K_{\ell+1}(m)$ for some $m \geq 0$.*

Let $\mathcal{C}_{\geq m}$ be the family of all cycles on at least m vertices, and $R(\mathcal{C}_{\geq m}, H)$ the smallest N such that any two colored K_N contains either a red-cycle in $\mathcal{C}_{\geq m}$ or a blue H. To prove Theorem 3, the following Ramsey-theoretic result is needed.

Lemma 5. *Let H be a graph. Let $\chi(H) = k$, $\epsilon_0 = 1/100k$. Suppose that $n \geq 3|H|$ and $t = \lfloor \epsilon_0^2 n \rfloor$, then*

$$R(\mathcal{C}_{\geq t}, H) \leq 2|H| + 0.1n.$$

By Lemma 4, any two-colored K_N can be partitioned into ℓ disjoint red-paths and a blue $K_{\ell+1}(m)$. Thus the main proof is based on the following items.

- If these ℓ red-paths are totally sufficiently long, we can find a blue H between red paths.
- Otherwise, the blue $K_{\ell+1}(m)$ is sufficiently large to contain a blue H.

Proof of Lemma 5. Suppose that $N = 2|H| + 0.1n = 2|H| + 10k\epsilon_0 n$. A red–blue edge-coloring of K_N generates a red graph R and a blue graph B on the same vertex set. We shall prove that if R contains no cycle in $\mathcal{C}_{\geq t}$, then B contains a blue H.

Let P be a red path in R. As shown in Fig. 1, we continuously choose disjoint segments $S_1, \ldots, S_k$ in P such that there are $\lfloor \epsilon_0^2 n \rfloor$ vertices of P between S_i and S_{i+1} for each $i \geq 1$.

Claim 1. *There is no red edge between each pair S_i and S_j.*

Proof of Claim 1. Otherwise, there would be a cycle of length at least $\epsilon_0^2 n$, which contradicts to the condition of maximality of r. $\diamond$

The reason for choosing $S_1, \ldots, S_k$ is that if $|P| \geq |H| + k\epsilon_0^2 n$, then we can make $|S_i| = a_i$ so the claimed blue H in B comes from Claim 1.

For vertices u and v in a path P, denote by $Dist_P(u, v)$ the distance between u and v along P.