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Double covers of symplectic dual polar graphs

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ABSTRACT

Let $\Gamma = \Gamma(2n, q)$ be the dual polar graph of type $Sp(2n, q)$. Underlying this graph is a $2n$-dimensional vector space V over a field $\mathbb{F}_q$ of odd order q, together with a symplectic (i.e. nondegenerate alternating bilinear) form $B : V \times V \to \mathbb{F}_q$. The vertex set of Γ is the set $\mathcal{V}$ of all n-dimensional totally isotropic subspaces of V. If $q \equiv 1 \bmod 4$, we obtain from Γ a nontrivial two-graph $\Delta = \Delta(2n, q)$ on $\mathcal{V}$ invariant under $PSp(2n, q)$. This two-graph corresponds to a double cover $\widehat{\Gamma} \to \Gamma$ on which is naturally defined a Q-polynomial $(2n + 1)$-class association scheme on $2|\widehat{\mathcal{V}}|$ vertices.

1. Introduction

Association schemes [2,6] were first defined by Bose and Mesner [3] in the context of the design of experiments. Philippe Delsarte used association schemes to unify the study of coding theory and design theory in his thesis [8], where he derived his well-known linear programming bound which has since found many applications in combinatorics. There he identified two types of association schemes which were of particular interest: the so-called P-polynomial and Q-polynomial schemes. Schemes which are P-polynomial are precisely those arising from distance-regular graphs, and are well studied. In particular, much effort has gone into the classification of distance-transitive graphs, the P-polynomial schemes which are the orbitals of a permutation group; and it is likely that all such examples are known. Also well-studied are the schemes which are both Q-polynomial and P-polynomial. A well-known conjecture [2, p. 312] of Bannai and Ito is the following: for sufficiently large d, a primitive scheme is P-polynomial if and only if it is Q-polynomial.

Classification efforts for Q-polynomial schemes are far less advanced than in the P-polynomial case; in particular it is likely that more examples from permutation groups are yet to be found. The Q-polynomial property has no known combinatorial characterization, making their study more difficult. However, the list of known examples (see [12,14,19]) indicates that these objects have interesting structure from the viewpoint of designs, lattices, coding theory and finite geometry.

In this paper, we give a new family of imprimitive Q-polynomial schemes with an unbounded number of classes. These schemes are formed by the orbitals of a group, giving a double cover of the scheme arising from the symplectic dual polar space graph. We note that only one other family of imprimitive Q-polynomial schemes with an unbounded number of classes is known that is not P-polynomial, namely the bipartite doubles of the Hermitian dual polar space graphs, which are Q-bipartite and Q-antipodal. The schemes in this paper are Q-bipartite, and have two Q-polynomial orderings. Except when the field order q is a square, the splitting field of these schemes is also irrational. We note that this is the only known

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family of Q-polynomial schemes with unbounded number of classes and an irrational splitting field. In the last section we give open parameters for hypothetical primitive Q-polynomial subschemes of this family.

Our paper is organized as follows: Background material on Gaussian coefficients, two-graphs and double covers of graphs, are covered in Sections 2–3. In Section 4 we recall the standard construction of the symplectic dual polar graph $\Gamma = \Gamma(2n, q)$. There we also introduce the Maslov index, which we use in Section 5 to construct the double cover $\widehat{\Gamma} \to \Gamma$ when $q \equiv 1 \bmod 4$. In Section 6 we construct a $(2n+1)$-class association scheme $\mathcal{S} = \mathcal{S}_{n,q}$ from $\widehat{\Gamma}$; and in Section 7 we show that $\mathcal{S}$ is Q-polynomial. The P-matrix of the scheme is constructed in Section 8. A particularly tantalizing open problem is the question whether $\mathcal{S}$ is in general the extended Q-bipartite double of a primitive Q-polynomial scheme; see Section 9.

2. Gaussian coefficients

For all integers n, k we define the *Gaussian coefficient*

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \dfrac{(q^n-1)(q^{n-1}-1)\cdots(q^{n-k+1}-1)}{(q^k-1)(q^{k-1}-1)\cdots(q-1)}, & \text{if } k \geqslant 0; \\ 0, & \text{if } k < 0. \end{cases}$$

In particular for $k = 0$ the empty product gives $\left[\begin{smallmatrix} n \\ 0 \end{smallmatrix}\right] = 1$. In later sections, q will be a fixed prime power; but here we may regard q as an indeterminate, so that for $n \geqslant 0$, after canceling factors we find $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right] \in \mathbb{Z}[q]$; and specializing to $q = 1$ gives the ordinary binomial coefficients $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_1 = \binom{n}{k}$. For general $n \in \mathbb{Z}$ we instead obtain a Laurent polynomial in q with integer coefficients, i.e. $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right] \in \mathbb{Z}[q, q^{-1}]$, as follows from conclusion (ii) of the following.

Proposition 2.1. *Let $n, k, \ell \in \mathbb{Z}$. The Gaussian coefficients satisfy*

(i) $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right] = q^k \left[\begin{smallmatrix} n-1 \\ k \end{smallmatrix}\right] + \left[\begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} n-1 \\ k \end{smallmatrix}\right] + q^{n-k} \left[\begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix}\right]$;

(ii) $\left[\begin{smallmatrix} -n \\ k \end{smallmatrix}\right] = (-q^{-n})^k \left[\begin{smallmatrix} n+k-1 \\ k \end{smallmatrix}\right]$;

(iii) $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right] \left[\begin{smallmatrix} k \\ \ell \end{smallmatrix}\right] = \left[\begin{smallmatrix} n \\ \ell \end{smallmatrix}\right] \left[\begin{smallmatrix} n-\ell \\ k-\ell \end{smallmatrix}\right]$;

(iv) $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right] = \left[\begin{smallmatrix} n \\ n-k \end{smallmatrix}\right]$ *whenever* $0 \leqslant k \leqslant n$. □

Most of the conclusions of Proposition 2.1 are found in standard references such as [1]. However, our definition of $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$ differs from the standard definition found in most sources, which either leave $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$ undefined for $n < 0$, or define it to be zero in that case. Our extension to all $n \in \mathbb{Z}$ means that the recurrence formulas (i) hold for all integers n, k, unlike the 'standard definition' which fails for $n = k = 0$. Property (i) plays a role in our later algebraic proofs using generating functions. In further defense of our definition, we observe that it has become standard to extend the definition of binomial coefficients $\binom{n}{k}$ so that $\binom{-n}{k} = (-1)^k \binom{n+k-1}{k}$ (see e.g. [1, p. 12]); and (ii) naturally generalizes this to Gaussian coefficients. We further note that (iii) holds for all $n, k \in \mathbb{Z}$ whether one takes the standard definition of $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$ or ours. The one advantage of the standard definition is that it renders superfluous the extra restriction $0 \leqslant k \leqslant n$ in the symmetry condition (iv). The interpretation of $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$ as the number of k-subspaces of an n-space over $\mathbb{F}_q$ is valid for all $n \geqslant 0$.

In Section 8 we will make use of the well-known generating polynomials

$$E_m(t) = \prod_{i=0}^{m-1}(1+q^i t) = \sum_{\ell=0}^{\infty} q^{\binom{\ell}{2}} \begin{bmatrix} m \\ \ell \end{bmatrix} t^\ell \quad \text{for } m = 0, 1, 2, \ldots;$$

note that in the latter sum, the terms for $\ell > m$ vanish, yielding $E_m(t) \in \mathbb{Z}[q, t]$ (or after specializing to a fixed prime power q, we obtain $E_m(t) \in \mathbb{Z}[t]$). Here we see the usual binomial coefficient $\binom{\ell}{2} = \frac{1}{2}\ell(\ell-1)$. In Section 8 we will make use of the following obvious relations:

Proposition 2.2. *For all $m \geqslant 0$, the generating function $E_m(t)$ satisfies*

(i) $E_m(-qt) = \frac{1-q^m t}{1-t} E_m(-t)$;

(ii) $E_m(q^2 t) = \frac{1+q^{m+1} t}{1+qt} E_m(qt)$; *and*

(iii) $E_m(r^3 t) = \frac{1+rq^m t}{1+rt} E_m(rt)$ *where* $r = \sqrt{q}$. □

3. Two-graphs and double covers of graphs

Here we describe the most basic connections between two-graphs and double covers of graphs; see [13,15,6,17] for more details. Our notation is chosen to conform to that used in subsequent sections.

Let $\mathcal{V}$ be any set. Denote by $\binom{\mathcal{V}}{k}$ the collection of all k-subsets of $\mathcal{V}$ (i.e. subsets of cardinality k). A *two-graph on* $\mathcal{V}$ is a subset $\Delta \subseteq \binom{\mathcal{V}}{3}$ such that for every 4-set $\{x, y, z, w\} \in \binom{\mathcal{V}}{4}$, an even number, i.e. 0, 2 or 4, of the triples $\{x, y, z\}, \{x, y, w\}, \{x, z, w\}$,