Discrete Mathematics 339 (2016) 597–605

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Weakly quasi-Hamiltonian-connected multipartite tournaments

Michel Surmacs

Lehrstuhl C für Mathematik, RWTH Aachen University, 52062 Aachen, Germany

ARTICLE INFO

Article history:
Received 15 September 2014
Received in revised form 22 April 2015
Accepted 3 October 2015
Available online 11 November 2015

Keywords:
Multipartite tournament
Quasi-Hamiltonian connectivity

ABSTRACT

A multipartite or c-partite tournament is an orientation of a complete c-partite graph. In 2013, Lu, Guo and Surmacs introduced the concept of quasi-Hamiltonian paths, that is to say, a directed path containing vertices from each partite set, in multipartite tournaments. They established that every 4-strong multipartite tournament is strongly quasi-Hamiltonian-connected – i.e., for every pair of vertices x_1, x_2, there is a quasi-Hamiltonian path from x_1 to x_2.

In this paper, we characterize all weakly quasi-Hamiltonian-connected multipartite tournaments – i.e., for every pair of vertices, there is at least one quasi-Hamiltonian path between them. Our results include and extend corresponding ones concerning tournaments due to Thomassen.

1. Introduction and terminology

We refer to Bang-Jensen and Gutin [4] for terminology not explicitly introduced here. A *multipartite* or *c-partite tournament* is a digraph $D = (V, A)$, where A is the arc set of D and the *vertex set* V is the union of c pairwise disjoint *partite sets* $V_1, \ldots, V_c$ such that there are no arcs between vertices from the same and exactly one arc between vertices from distinct partite sets. To determine the partite set of each vertex, we define the function

$$\mathrm{p} : V \to \{1, \ldots, c\}, \quad x \mapsto \mathrm{p}(x) :\Leftrightarrow x \in V_{\mathrm{p}(x)}.$$

For a subset $X \subseteq V$, $V_{\mathrm{p}(X)}$ is the union $\bigcup_{x \in X} V_{\mathrm{p}(x)}$ of partite sets represented in X. For convenience, we also allow a subdigraph D' of D in our notation instead of its vertex set.

For an arbitrary digraph D, its vertex set and arc set are denoted by $V(D)$ and $A(D)$, respectively. Instead of $(x, y) \in A(D)$, we mostly use $xy \in A(D)$.

Let $X, Y \subseteq V(D)$ be two disjoint subsets and note that, if X or Y consist of a single vertex, we omit the braces in all following notation. Then $X \to Y$ ($X \Rightarrow Y$, respectively) denotes that there is an arc from every vertex in X to every vertex in Y (no arc from a vertex in Y to a vertex in X, respectively). In the former case, we also say that X *dominates* Y. For subdigraphs D_1 and D_2 of D on disjoint vertex sets, we also write $D_1 \to D_2$ and $D_1 \Rightarrow D_2$ to express the respective property of their vertex sets.

Furthermore, $(X, \{xy \mid x, y \in X, xy \in A(D)\})$ is the subgraph of D *induced by* X, denoted as $D[X]$. $D - X$ denotes the subgraph $D[V(D)\backslash X]$. For an arc xy of the complete digraph on $V(D)$, we define $D + xy$ as $(V, A(D) \cup \{xy\})$. The *out-neighborhood* $N^+_{D[X]}(x)$ of a vertex $x \in X$ is defined as $\{y \mid xy \in A(D[X])\}$. The *in-neighborhood* $N^-_{D[X]}(x)$ is defined accordingly.

An (x_1, x_{l+1})-*path of length* l, also called an *l-path from* x_1 *to* x_{l+1}, is a sequence $P = x_1 \ldots x_{l+1}$ of pairwise distinct vertices such that $x_i x_{i+1} \in A(D)$ holds for all $i \in \{1, \ldots, l\}$. An *l-cycle* is defined analogously with the only distinction being that its

E-mail address: michel.surmacs@rwth-aachen.de.

http://dx.doi.org/10.1016/j.disc.2015.10.003
0012-365X/

initial and its terminal vertex are identical. A path (cycle, respectively) in D is called *Hamiltonian*, if it contains all vertices of D. For two paths P and Q on disjoint vertex sets such that the terminal vertex of P dominates the initial vertex of Q, we write PQ to denote the path we obtain by appending Q to P.

A *strong component* D' of D is a maximal induced subdigraph such that there is an (x, y)-path for all $x, y \in V(D')$. A digraph is called *strong*, if it consists of exactly one strong component. It is called *k-strong*, if $|V(D)| \geq k + 1$ and $D - U$ is strong for all subsets $U \subseteq V(D)$ with $|U| < k$.

Note that the following useful decomposition – we call it the *multipartite decomposition* of D – was originally stated by Tewes and Volkmann only for connected, non-strong multipartite tournaments. Obviously, strong multipartite tournaments, as well as those consisting of only one partite set, are also covered by the case $r = 1$.

Lemma 1.1 ([13]). *Let D be a c-partite tournament with partite sets $V_1, \ldots, V_c$. Then there exists a unique decomposition of $V(D)$ into pairwise disjoint subsets $X_1, \ldots, X_r$, where X_i is the vertex set of a strong component of D or $X_i \subseteq V_l$ for some $l \in \{1, \ldots, c\}$ such that $X_i \Rightarrow X_j$ for $1 \leq i < j \leq r$ and there are $x_i \in X_i$ and $y_i \in X_{i+1}$ such that $x_i \rightarrow y_i$ for $1 \leq i < r$.*

Analogous to the index of the partite set of a vertex, we define the *component number*-function

$$\mathrm{cn}_D : V(D) \rightarrow \{1, \ldots, r\}, \quad v \mapsto \mathrm{cn}_D(v) :\Leftrightarrow v \in X_{\mathrm{cn}_D(v)}$$

to determine the index of its component.

Tournaments and their generalizations are arguably the best studied classes of digraphs, as documented by several surveys (see, e.g., [9,12,5,1–3]) on the matter, published over the course of the past decades. For a survey of results on multipartite tournaments in particular, see for example [15]. As interest shifted towards superclasses of tournaments, obviously, one natural aim is to extend known results on tournaments to the generalized tournaments. However, frequently, the process of generalizing such results is not a straight forward one, as can be observed, for instance, using the example of Moon's theorem, one of the first and most central results on tournaments.

Theorem 1.2 ([11]). *Every vertex of a strong tournament T is contained in an l-cycle for all $l \in \{3, \ldots, |V(T)|\}$.*

As early as 1976, Bondy [6] gave examples of strong c-partite tournaments on $n > c$ vertices without cycles whose length exceeds c. To extend Moon's theorem to multipartite tournaments, we therefore have to adapt it to the new structure of the superclass. There have been several successful attempts of such a generalization (see, e.g., [15]), each with their own distinct perspective. The following particular one is due to Goddard and Oellermann.

Theorem 1.3 ([7]). *Every vertex of a strong c-partite tournament belongs to a cycle that contains vertices from exactly m partite sets for each $m \in \{3, 4, \ldots, c\}$.*

Since partite sets of tournaments consist of a singular vertex, Theorem 1.3 obviously includes and extends Moon's theorem. Instead of the traditional length – i.e., the number of vertices contained in a cycle – Goddard and Oellermann consider a sort of quasi-length – i.e., the number of distinct partite sets represented in the cycle. In the tournament setting, the definitions are equivalent, but in the superclass of multipartite tournaments, the change in perspective allows for the generalized result.

This approach inspired Lu, Guo and Surmacs [10] to introduce the idea of *quasi-Hamiltonian* paths and cycles, which – instead of containing all vertices – contain vertices from every partite set of a multipartite tournament. Accordingly, two distinct vertices $x, y \in V(D)$ are *weakly* (*strongly*, respectively) *quasi-Hamiltonian-connected*, if there exists a quasi-Hamiltonian (x, y)-path or (and, respectively) a quasi-Hamiltonian (y, x)-path in D. A multipartite tournament D is called *weakly* (*strongly*, respectively) *quasi-Hamiltonian-connected* if each pair of distinct vertices is weakly (strongly, respectively) quasi-Hamiltonian-connected. Note that, again, quasi-Hamiltonian connectivity corresponds to Hamiltonian connectivity in tournaments. Furthermore, Lu, Guo and Surmacs established the following theorem.

Theorem 1.4 ([10]). *Every 4-strong multipartite tournament is strongly quasi-Hamiltonian-connected.*

Using this idea, in this paper, we generalize the following characterization of weakly Hamiltonian-connected tournaments due to Thomassen.

Theorem 1.5 ([14]). *A tournament T with at least three vertices is weakly Hamiltonian-connected, if and only if it satisfies* (i), (ii) *and* (iii) *below.*

(i) *T is strong.*
(ii) *For each vertex x of T, $T - x$ has at most two strong components.*
(iii) *$T \ncong T_6^+$ and $T \ncong T_6^-$ [see definition below].*

To that end, we introduce four classes of non-weakly-quasi-Hamiltonian-connected multipartite tournaments. First, we need some additional notation. Fig. 1 depicts two examples of non-weakly-quasi-Hamiltonian-connected multipartite tournaments which are also essential for the definitions following it.