Discrete Mathematics 339 (2016) 646–657

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

A non-trivial intersection theorem for permutations with fixed number of cycles

Cheng Yeaw Ku [a,*], Kok Bin Wong [b]

[a] Department of Mathematics, National University of Singapore, Singapore 117543, Singapore

[b] Institute of Mathematical Sciences, University of Malaya, 50603 Kuala Lumpur, Malaysia

ARTICLE INFO

Article history:
Received 21 January 2015
Received in revised form 16 July 2015
Accepted 28 September 2015
Available online 11 November 2015

Keywords:
t-intersecting family
Frankl–Hilton–Milner
Erdős–Ko–Rado
Permutations

ABSTRACT

Let S_n denote the set of permutations of $[n] = \{1, 2, \ldots, n\}$. For a positive integer k, define $S_{n,k}$ to be the set of all permutations of $[n]$ with exactly k disjoint cycles, i.e.,

$$S_{n,k} = \{\pi \in S_n : \pi = c_1 c_2 \cdots c_k\},$$

where $c_1, c_2, \ldots, c_k$ are disjoint cycles. The size of $S_{n,k}$ is given by $\begin{bmatrix} n \\ k \end{bmatrix} = (-1)^{n-k} s(n,k)$, where $s(n,k)$ is the Stirling number of the first kind. A family $\mathcal{A} \subseteq S_{n,k}$ is said to be t-cycle-intersecting if any two elements of $\mathcal{A}$ have at least t common cycles. A family $\mathcal{A} \subseteq S_{n,k}$ is said to be trivially t-cycle-intersecting if $\mathcal{A}$ is the stabiliser of t fixed points, i.e., $\mathcal{A}$ consists of all permutations in $S_{n,k}$ with some t fixed cycles of length one. For $1 \le j \le t$, let

$$\mathcal{Q}_j = \left\{\sigma \in S_{n,k} \ : \ \sigma(i) = i \text{ for all } i \in [k] \setminus \{j\}\right\}.$$

For $t + 1 \le s \le k$, let

$$\mathcal{B}_s = \left\{\sigma \in S_{n,k} \ : \ \sigma(i) = i \text{ for all } i \in [t] \cup \{s\}\right\}.$$

In this paper, we show that, given any positive integers k, t with $k \ge 2t + 3$, there exists an $n_0 = n_0(k,t)$, such that for all $n \ge n_0$, if $\mathcal{A} \subseteq S_{n,k}$ is non-trivially t-cycle-intersecting, then

$$|\mathcal{A}| \le |\mathcal{B}|,$$

where $\mathcal{B} = \bigcup_{s=t+1}^{k} \mathcal{B}_s \cup \bigcup_{j=1}^{t} \mathcal{Q}_j$. Furthermore, equality holds if and only if $\mathcal{A}$ is a conjugate of $\mathcal{B}$, i.e., $\mathcal{A} = \beta^{-1}\mathcal{B}\beta$ for some $\beta \in S_n$.

1. Introduction

Let $[n] = \{1, \ldots, n\}$, and let $\binom{[n]}{k}$ denote the family of all k-subsets of $[n]$. A family $\mathcal{A}$ of subsets of $[n]$ is *t-intersecting* if $|A \cap B| \ge t$ for all $A, B \in \mathcal{A}$. One of the most beautiful results in extremal combinatorics is the Erdős–Ko–Rado theorem.

* Corresponding author.
E-mail addresses: matkcy@nus.edu.sg (C.Y. Ku), kbwong@um.edu.my (K.B. Wong).

http://dx.doi.org/10.1016/j.disc.2015.09.032
0012-365X/

Theorem 1.1 (*Erdős, Ko, and Rado [14], Frankl [15], Wilson [46]*). *Suppose $\mathcal{A} \subseteq \binom{[n]}{k}$ is t-intersecting and $n > 2k - t$. Then for $n \geq (k - t + 1)(t + 1)$, we have*

$$|\mathcal{A}| \leq \binom{n-t}{k-t}.$$

Moreover, if $n > (k - t + 1)(t + 1)$ then equality holds if and only if $\mathcal{A} = \{A \in \binom{[n]}{k} \ : \ T \subseteq A\}$ for some t-set T.

In the celebrated paper [2], Ahlswede and Khachatrian extended the Erdős–Ko–Rado theorem by determining the structure of all t-intersecting set systems of maximum size for all possible n (see also [4,18,26,31,37,40,42,43,45] for some related results). There have been many recent results showing that a version of the Erdős–Ko–Rado theorem holds for combinatorial objects other than set systems. For example, an analogue of the Erdős–Ko–Rado theorem for the Hamming scheme is proved in [41]. A complete solution for the t-intersection problem in the Hamming space is given in [3]. Intersecting families of permutations were initiated by Deza and Frankl in [11]. Some recent work done on this problem and its variants can be found in [6,8,9,12,13,20,27,29,38,39,44]. The investigation of the Erdős–Ko–Rado property for graphs started in [24], and gave rise to [5,7,22,23,25,47]. The Erdős–Ko–Rado type results also appear in vector spaces [10,19], set partitions [28,32,30] and weak compositions [33,35,36].

For a family $\mathcal{A}$ of k-subsets, $\mathcal{A}$ is said to be *trivially* t-intersecting if there exists a t-set $T = \{x_1, \dots, x_t\}$ such that all members of $\mathcal{A}$ contain T. The Erdős–Ko–Rado theorem implies that a t-intersecting family of maximum size must be trivially t-intersecting when n is sufficiently large in terms of k and t.

Hilton and Milner [21] proved a strengthening of the Erdős–Ko–Rado theorem for $t = 1$ by determining the maximum size of a non-trivial 1-intersecting family. A short and elegant proof was later given by Frankl and Füredi [17] using the shifting technique.

Theorem 1.2 (*Hilton-Milner*). *Let $\mathcal{A} \subseteq \binom{[n]}{k}$ be a non-trivial 1-intersecting family with $k \geq 4$ and $n > 2k$. Then*

$$|\mathcal{A}| \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1.$$

Equality holds if and only if

$$\mathcal{A} = \left\{X \in \binom{[n]}{k} : x \in X, X \cap Y \neq \emptyset\right\} \cup \{Y\}$$

for some k-subset $Y \in \binom{[n]}{k}$ and $x \notin Y$.

Frankl [16] proved the following theorem which is a generalisation of Theorem 1.2.

Theorem 1.3 (*Frankl*). *Given any positive integers k, t with $k \geq 2t + 2$, there exists a constant $n_0(k, t)$ depending only on k and t, such that for all $n \geq n_0(k, t)$, if $\mathcal{A} \subseteq \binom{[n]}{k}$ is non-trivially t-intersecting then*

$$|\mathcal{A}| \leq |\mathcal{W}_0|,$$

where

$$\mathcal{W}_0 = \left\{A \in \binom{[n]}{k} \ : \ [t] \subseteq A, A \cap \{t+1, \dots, k+1\} \neq \emptyset\right\} \cup \{[k+1] \setminus \{i\} \ : \ i \in [t]\}.$$

Furthermore, equality holds if and only if $\mathcal{A} = \beta \mathcal{W}_0$ for some $\beta \in S_n$.

In fact, the complete result on non-trivial intersection problems for finite sets was obtained by Ahlswede and Khachatrian in their seminal paper [1].

Let S_n denote the set of permutations of $[n]$. For a positive integer k, define $S_{n,k}$ to be the set of all permutations of $[n]$ with exactly k disjoint cycles, i.e.,

$$S_{n,k} = \{\pi \in S_n : \pi = c_1 c_2 \cdots c_k\},$$

where $c_1, c_2, \dots, c_k$ are disjoint cycles. It is well known that the size of $S_{n,k}$ is given by $\left[{n \atop k}\right] = (-1)^{n-k} s(n, k)$, where $s(n, k)$ is the *Stirling number of the first kind*.

We shall use the following notations:

(a) $N(c) = \{a_1, a_2, \dots, a_l\}$ for a cycle $c = (a_1, a_2, \dots, a_l)$;
(b) $M(\pi) = \{c_1, c_2, \dots, c_k\}$ for a $\pi = c_1 c_2 \dots c_k \in S_{n,k}$.