Discrete Mathematics 339 (2016) 658–667

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Small bi-regular graphs of even girth

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ARTICLE INFO

Article history:
Received 21 November 2013
Received in revised form 3 April 2015
Accepted 6 October 2015
Available online 11 November 2015

Keywords:
Cage
Bi-regular cage
Recursive construction
Order

ABSTRACT

A graph of girth g that contains vertices of degrees r and m is called a bi-regular $(\{r, m\}, g)$-graph. As with the *Cage Problem*, we seek the smallest $(\{r, m\}, g)$-graphs for given parameters $2 \leq r < m, g \geq 3$, called $(\{r, m\}, g)$-cages. The orders of the majority of $(\{r, m\}, g)$-cages, in cases where m is much larger than r and the girth g is odd, have been recently determined via the construction of an infinite family of graphs whose orders match a well-known lower bound, but a generalization of this result to bi-regular cages of even girth proved elusive.

We summarize and improve some of the previously established lower bounds for the orders of bi-regular cages of even girth and present a generalization of the odd girth construction to even girths that provides us with a new general upper bound on the order of graphs with girths of the form $g = 2t$, t odd. This construction produces infinitely many $(\{r, m\}; 6)$-cages with sufficiently large m. We also determine a $(\{3, 4\}; 10)$-cage of order 82.

1. Preliminaries

The concept of bi-regular cages has been introduced in hopes of shedding some new light on the notoriously hard *Cage Problem* — the problem of determining the smallest possible orders of k-regular graphs of girth g for $k \geq 2, g \geq 3$. The two problems share a number of characteristics. The existence of (k, g)-graphs for any pair (k, g) with $k \geq 2$ and $g \geq 3$ has been established by Erdős and Sachs in [10,6]. A parallel result asserting the existence of $(\{r, m\}; g)$-graphs for any set of parameters $2 \leq r < m$ and $g \geq 3$ has been shown by Chartrand, Gould and Kapoor in [4]. Similarly, a lower bound on the order of bi-regular graphs in terms of their degrees and girths can be obtained based on the same intuitive counting argument as the well-known Moore bound for the order of $n(k, g)$-cages [5]: for $2 \leq r < m$ and $g \geq 3$,

$$n(\{r, m\}; g) \geq \begin{cases} 1 + m\sum_{i=0}^{t-1}(r-1)^i, & \text{for } g = 2t+1, \\ 1 + m\sum_{i=0}^{t-2}(r-1)^i + (r-1)^{t-1}, & \text{for } g = 2t. \end{cases} \tag{1}$$

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http://dx.doi.org/10.1016/j.disc.2015.10.009
0012-365X/

However, unlike the case of the Moore bound for regular cages – which is known to be sharp for only a few families of parameters (k, g) (see, for example, [7]) – the above lower bound for bi-regular cages has been proved sharp for almost all bi-regular graphs of *odd* girth:

Theorem 1.1 *([8]). For every $r \geq 3$ and every odd $g = 2t + 1 \geq 3$, there exists an integer m_0 such that for every even $m \geq m_0$, the bi-regular $(\{r, m\}, g)$-cage is of order*

$$1 + \sum_{i=1}^{t} m(r-1)^{i-1}.$$

In addition, when r is odd, the restriction on the parity of m can be removed, and there exists an integer m_0 such that a bi-regular $(\{r, m\}, g)$-cage of the above order exists for all $m \geq m_0$.

The essence of this proof lies in a construction that adds edges (and only edges) to a tree with the number of vertices matching (1). In addition, even though the degrees m obtained in the proof of this result in [8] are much larger than the corresponding degrees r, computational evidence seems to suggest the existence of bi-regular graphs of odd girth and order equal to the lower bound (1) starting already from m's differing from r by 1 or 2. It is also interesting to note that all graphs of order matching the lower bound (1) constructed in [8] have the property that all but one of their vertices are of degree r — a clear indication that, in the case of odd girth, allowing for even just one vertex of higher degree makes the problem of finding bi-regular cages significantly easier than the original cage problem.

The case of bi-regular cages of even girth bigger than 4 appears to be more complicated.[1] This is mainly due to the fact that the intuitive lower bound (1) has been shown to be strictly smaller than the order of the $(\{r, m\}, 2t)$-cages, for all $t \geq 3$ [11,1]. As this can be also shown using ideas we use in the proofs throughout this paper, we reprove this result for illustration:

Lemma 1.2. *Let $\mathcal{G}$ be an $(\{r, m\}; g)$-graph of even girth $g = 2t \geq 6$. Then*

$$|V(\mathcal{G})| > 1 + m\sum_{i=0}^{t-2}(r-1)^i + (r-1)^{t-1}.$$

Proof. We proceed by contradiction. Let $\mathcal{G}$ be an $(\{r, m\}; g)$-graph, $g = 2t \geq 6$, and suppose that $|V(\mathcal{G})| = 1 + m\sum_{i=0}^{t-2}(r-1)^i + (r-1)^{t-1}$ (note that this is the value from the lower bound (1), so $|V(\mathcal{G})|$ is not smaller than this expression). Then $\mathcal{G}$ contains at least one vertex u of degree m, and the subgraph $\mathcal{G}_u$ of $\mathcal{G}$ induced by the set of vertices of $\mathcal{G}$ of distance not larger than $t-1$ must be a tree as otherwise we would violate the girth $g = 2t$ of $\mathcal{G}$. Since u is of degree m, and every vertex of $\mathcal{G}_u$ that is not a leaf is of degree at least r, $|V(\mathcal{G}_u)| \geq 1 + m\sum_{i=0}^{t-2}(r-1)^i$, and the number of edges incident to the leaves of $\mathcal{G}_u$ is at least $m(r-1)^{t-1}$. Note that these edges all terminate in $V(\mathcal{G}) - V(\mathcal{G}_u)$. By the assumption about the order of $\mathcal{G}$, the set of vertices that do not belong to $\mathcal{G}_u$ is of size $(r-1)^{t-1}$. This can only happen if all the above edges terminate in vertices of degree m, i.e., the set $V(\mathcal{G}) - V(\mathcal{G}_u)$ consists entirely of vertices of order m (each of them incident with exactly m edges from the set of edges incident to the leaves of $\mathcal{G}_u$). Thus, $\mathcal{G}$ contains at least two vertices of degree m that are of distance 2 in $\mathcal{G}$. Let us assume without loss of generality that one of these vertices is the vertex u and the other one is v. Since $t \geq 3$, $v \in V(\mathcal{G}_u)$ and therefore $\mathcal{G}_u$ contains more vertices than just $1 + m\sum_{i=0}^{t-2}(r-1)^i$. In addition, the number of edges incident to the branch of $\mathcal{G}_u$ that contains v is greater than $(r-1)^{t-1}$. Furthermore, none of the edges attached to leaves of this branch can be attached to the same vertex as that would violate the girth of $\mathcal{G}$. Thus,

$$|V(\mathcal{G})| = |V(\mathcal{G}_u)| + |V(\mathcal{G}) - V(\mathcal{G}_u)| > 1 + m\sum_{i=0}^{t-2}(r-1)^i + (r-1)^{t-1}. \quad \square$$

As argued in the above lemma, the bound (1) is universally unachievable. In addition, the forthcoming lower bounds as well as computational evidence suggest that the orders of bi-regular cages of even girth are quite bigger than (1) — the situation with bi-regular cages of even girth appears to be parallel to that with the original regular cages.

We begin the list of the improved lower bounds on the order of even-girth bi-regular cages with girth 6. First, the following lower bound has been proved in [11] for all $3 \leq r < m$:

$$n(\{r, m\}; 6) \geq 2(rm - m + 1). \tag{2}$$

This bound has been shown to be sharp for all $(\{3, m\}; 6)$-cages with $m > 3$ in [9], for all $(\{r, m\}; 6)$-cages with $2 \leq r \leq 5$ and $m > r$, as well as for all $(\{r, m\}; 6)$-cages with $m - 1$ a prime power and $2 \leq r < m$ in [11], and finally, for $r - 1$ a prime power and all $(\{r, kr\}; 6)$-cages, $k \geq 2$, in [3]. In addition, it was conjectured in [11], that the bound is sharp for all

[1] Note for completeness that for girth 4 the order of the cages matches the lower bound (1), $n(\{r, m\}; 4) = r + m$, with the cages being the complete bipartite graphs $K_{r,m}$.