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Note

Spanning trees homeomorphic to a small tree

Akira Saito *, Kazuki Sano

Department of Information Science, Nihon University, Sakurajosui 3–25–40, Setagaya-Ku, Tokyo 156–8550, Japan

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ABSTRACT

A classical result of Ore states that if a graph G of order n satisfies $\deg_G x + \deg_G y \geq n - 1$ for every pair of nonadjacent vertices x and y of G, then G contains a hamiltonian path. In this note, we interpret a hamiltonian path as a spanning tree which is a subdivision of K_2 and extend Ore's result to a sufficient condition for the existence of a spanning tree which is a subdivision of a tree of a bounded order. We prove that for a positive integer k, if a connected graph G satisfies $\deg_G x + \deg_G y \geq n - k$ for every pair of nonadjacent vertices x and y of G, then G contains a spanning tree which is a subdivision of a tree of order at most $k + 2$. We also discuss the sharpness of the result.

1. Introduction

There are many studies on spanning trees which are inspired by a hamiltonian path. They interpret a hamiltonian path as a spanning tree with an additional property and take a certain sufficient condition for the existence of a hamiltonian path. Then by relaxing the condition, they observe how this additional property changes. There are several different views on the additional property. For example, a hamiltonian path is a spanning tree of maximum degree at most two. This fact leads us to the notion of a k-*tree*, which is a spanning tree of maximum degree at most a given constant k. Another study interprets a hamiltonian path as a spanning tree with two leaves, where a leaf of a tree T is a vertex of degree at most one in T. We can generalize this interpretation to the notion of a k-*ended tree*, which is a spanning tree with at most k leaves. Both k-trees and k-ended trees have been investigated in a number of papers. To the readers who are interested in these topics, we refer the recent survey by Ozeki and Yamashita [5].

In this note, we take a different approach. A path of order at least two is a subdivision of K_2. Motivated by this observation, we investigate a sufficient condition for a graph to contain a spanning tree which is homeomorphic to a tree of a bounded order.

Let k be a positive integer and let G be a graph. If $k \leq \alpha(G)$, where $\alpha(G)$ is the independence number of G, we define $\sigma_k(G)$ by

$$\sigma_k(G) = \min\left\{\sum_{x\in S} \deg_G x \colon S \text{ is an independent set of } G \text{ of order } k\right\}.$$

If $k > \alpha(G)$, we define $\sigma_k(G) = +\infty$. Ore [4] has proved that for an integer n with $n \geq 3$, a graph G of order n with $\sigma_2(G) \geq n$ contains a hamiltonian cycle. As an easy corollary of this result, we obtain the following theorem.

Theorem A. *A graph G of order n with $\sigma_2(G) \geq n - 1$ contains a hamiltonian path.*

* Corresponding author.
E-mail address: asaito@chs.nihon-u.ac.jp (A. Saito).

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The purpose of this note is to extend Theorem A and prove the following theorem.

Theorem 1. *Let k be a positive integer. Then a connected graph G of order n with $\sigma_2(G) \geq n-k$ contains a spanning tree which is homeomorphic to a tree of order at most $k+2$.*

Theorem A does not explicitly assume that G is connected since it is implied by $\sigma_2(G) \geq |V(G)|-1$. However, for $k \geq 2$, the condition $\sigma_2(G) \geq |V(G)|-k$ does not imply the connectedness of G. Therefore, we explicitly assume the connectedness of G in Theorem 1.

If we put $k=1$ in Theorem 1, the conclusion only guarantees the existence of a spanning tree homeomorphic to a tree of order at most three, which looks weaker than Theorem A. However, a tree of order three is a path and homeomorphic to K_2. Hence Theorem 1 actually implies Theorem A.

Seeing the discussion in the previous paragraph, one may suspect that under the same assumption as in Theorem 1, we can guarantee the existence of a spanning tree homeomorphic to a tree of order at most $k+1$. But this is not true for $k \geq 2$. We will discuss the sharpness of Theorem 1 in Section 3.

Broersma and Tuinstra [1] have proved the following theorem.

Theorem B (*Broersma and Tuinstra [1]*)**.** *Let k be a positive integer and let G be a connected graph of order n. If $\sigma_2(G) \geq n-k$, then G contains a $(k+1)$-ended tree.*

For $k \geq 1$, a tree homeomorphic to a tree of order at most $k+2$ contains at most $k+1$ leaves. Therefore, Theorem 1 implies Theorem B.

We give a proof of Theorem 1 in the next section, and we discuss the sharpness of Theorem 1 in Section 3. We make concluding remarks in Section 4.

For basic graph-theoretic notation and definitions not explained in this note, we refer the reader to [2]. Let T be a tree and let u and v be vertices in T. Then we denote by uTv the unique path from u and v in T. If u is an endvertex of a path P, we say that u and P are incident with each other. For a vertex x in a graph G, we denote by $N_G(x)$ the neighborhood of x in G. We say that G is nontrivial if $|V(G)| \geq 2$.

2. Proof of the main theorem

As we have mentioned in the introduction, a vertex of degree at most one in a tree T is called a *leaf*. On the other hand, we call a vertex of degree at least three in T a *branch vertex*. Let $L(T)$ and $S(T)$ be the sets of leaves and branch vertices of T, respectively.

Let G be a tree and let x be a vertex of degree two in G. Let $N_G(x)=\{u,v\}$ and assume $uv \notin E(G)$. Then the operation of deleting x and adding the edge uv is called *suppressing* x. It is a reverse operation of simple subdivision of the edge uv. If we successively suppress the vertices of degree two in a tree T, we eventually obtain a tree on $L(T) \cup S(T)$. We call this tree the *reduced tree* of T. Note that the reduced tree is uniquely determined, regardless of the order of the vertices chosen for suppression. Note also that the reduced tree does not have a vertex of degree two. Since every tree is a subdivision of its reduced tree, we can paraphrase Theorem 1 in the following way.

Theorem 2. *Let k and n be positive integers, and let G be a connected graph of order n. If $\sigma_2(G) \geq n-k$, then G has a spanning tree T with $|L(T)|+|S(T)| \leq k+2$.*

Let T be a tree of order at least two and let T_1 be its reduced tree. Then an edge of T_1 corresponds to a path in T which joins two vertices in $L(T) \cup S(T)$. A *bough* of T is a path in T corresponding to an edge of T_1 which is incident with a leaf. On the other hand, a path in T which corresponds to an edge of T_1 joining two branch vertices is called a *trunk* of T. Note that $E(T)$ is decomposed into the sets of edges of boughs and trunks of T.

We introduce a special branch vertex of a tree, which plays an important role in the proof of Theorem 2. Let T be a tree which contains at least one branch vertex. Let P be a bough of T and let z be the branch vertex of P that is incident with P. When we say that we delete P, we mean to delete $V(P)-\{z\}$ from T. Note that the resulting graph is a tree. The *pruned tree* of T is the tree obtained from T by deleting all the boughs of T. Let T' be the pruned tree of T. Then $L(T') \subset S(T)$. We call a member of $L(T')$ a *peripheral branch vertex* of T.

Let T be a tree with at least one branch vertex, and let z be a peripheral branch vertex of T. By the definition, if T contains two or more branch vertices, then exactly one trunk is incident with z, and the number of boughs incident with z is $\deg_T z - 1$. If z is the only branch vertex of T, then T has no trunk, and all the boughs of T are incident with z. In both cases, at least two boughs are incident with z.

Let G be a connected graph and let T be a spanning tree of G. If T is chosen so that

(1) $|L(T)|$ is as small as possible, and
(2) $|S(T)|$ is as small as possible, subject to (1),

then T is called an *optimal tree* of G.

We first make several observations about an optimal tree.