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Novel structures in Stanley sequences

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ABSTRACT

Given a set of integers with no 3-term arithmetic progression, one constructs a Stanley sequence by choosing integers greedily without forming such a progression. This paper offers two main contributions to the theory of Stanley sequences. First, we describe all known Stanley sequences with closed-form expressions as solutions to constraints in modular arithmetic, defining the *modular* and *pseudomodular Stanley sequences*. Second, we introduce the *basic Stanley sequences*, whose elements arise as the sums of finite subsets of a *basis* sequence. Applications of our results include the construction of Stanley sequences with arbitrarily large gaps between terms, answering a weak version of a problem by Erdős et al. Finally, we generalize several results about Stanley sequences to p-free sequences, where p is any odd prime.

1. Introduction

A set of nonnegative integers is said to be *p-free* if it contains no p-term arithmetic progressions. Much has been written about the maximum density of such a set [1–3,6–9,15,16,18–21]. Motivated by this, in 1978, Odlyzko and Stanley [12] proposed constructing 3-free sequences according to the following greedy algorithm.

Definition 1.1. Let $A = \{a_0, \dots, a_k\}$ be a 3-free set of non-negative integers satisfying $a_0 < \cdots < a_k$. We define the *Stanley sequence* $S(A) = (a_n)$ generated by A recursively as follows. If $a_0 < \cdots < a_n$ have already been defined for $n \geq k$, then a_{n+1} is the smallest positive integer greater than a_n such that $\{a_0, \dots, a_n, a_{n+1}\}$ is 3-free.

We will often write $S(a_0, \dots, a_k)$ for $S(\{a_0, \dots, a_k\})$. We will also sometimes consider the sequence $S(A)$ as a set.

The name "Stanley sequences" originates with Erdős et al. [5], who generalized the procedure above from the case of $|A| = 2$, as originally proposed by Odlyzko and Stanley. The simplest Stanley sequence is

$$S(0) = 0, 1, 3, 4, 9, 10, 12, 13, 27, \dots,$$

the elements of which are exactly those integers with no 2's in their ternary expansion; the growth rate of this sequence is given by $a_n = \Theta(n^{\log_2 3})$.

Remarkably, Stanley sequences appear to exhibit two distinct patterns of asymptotic growth [4,5,12], with no intermediate growth rate possible. The original conjecture of Odlyzko and Stanley was framed by Rolnick in [13, Conjecture 1.1] as follows.

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Conjecture 1.2. *Let $S(A) = (a_n)$ be a Stanley sequence. Then, for all n large enough, one of the following two patterns of growth is satisfied.*

- *Type* 1 : $\alpha/2 \le \liminf a_n/n^{\log_2 3} \le \limsup a_n/n^{\log_2 3} \le \alpha$, *or*
- *Type* 2 : $a_n = \Theta(n^2/\log n)$.

Odlyzko and Stanley [12] discovered that sequences of the form $S(0, 3^n)$ and $S(0, 2 \cdot 3^n)$ are of Type 1. Although [12] considered only the case $\alpha = 1$, Rolnick and Venkataramana have shown [14] that every rational number $\alpha \ge 1$ is possible for which the denominator is a power of 3. Odlyzko and Stanley showed that Type 2 growth is, in some sense, the "expected" growth of a Stanley sequence, assuming that elements occur in the sequence according to a continuous probability distribution. The formula $\Theta(n^2/\log n)$ has been experimentally verified by Lindhurst [10] up to large values of the sequence $S(0, 4)$. However, no Stanley sequence, including $S(0, 4)$, has been definitively proven to satisfy Type 2 growth. In this paper, we present a different approach to Type 1 sequences based upon modular arithmetic.

Erdős et al. posed several problems on the asymptotic behavior of Stanley sequences. In [11], Moy solved Problem 1 of [5] by proving that in any Stanley sequence (a_n), the terms a_n grow no faster than $n^2/(2+\epsilon)$, where ϵ is an arbitrary constant. An effective lower bound (Problem 2) remains open; that is, proving $\liminf \log a_n/\log n > 1$, where the lim inf is conjectured to be $\log_2 3$.

Erdős et al. also consider the gaps between consecutive elements, asking whether there exists a Stanley sequence (a_n) for which $\liminf(a_{n+1} - a_n) = \infty$ (Problem 4, [5]). A weaker version of this question (Problem 6) was answered in the affirmative by Savchev and Chen [17], who constructed a 3-free sequence (a_n) satisfying $\liminf(a_{n+1} - a_n) = \infty$ for which no integer can be added without violating the 3-free property; this sequence is however not a Stanley sequence. Among the results in this paper, we show that there exist Stanley sequences for which $\liminf(a_{n+1} - a_n)$ is arbitrarily large.

As noted in Erdős and Graham [4, page 22], sequences like $S(0, 4)$ seem to admit no closed-form description. However, Rolnick [13] has extensively studied sequences that exhibit Type 1 growth, constructing many sequences of this form. In particular, [13] introduces the concept of independent and regular Stanley sequences, which satisfy Type 1 growth. Rolnick conjectures that these are in fact the *only* Type 1 sequences; in this paper, however, we will show that the definitions must be slightly modified for this conjecture to hold.

Definition 1.3. A Stanley sequence $S(A) = (a_n)$ is *independent* if there exist constants $\lambda = \lambda(A)$ and $\kappa = \kappa(A)$ such that for all $k \ge \kappa$ and $0 \le i < 2^k$, we have

- $a_{2^k+i} = a_{2^k} + a_i$,
- $a_{2^k} = 2a_{2^k-1} - \lambda + 1$.

The constant λ is referred to as the *character*; it is proven in [13] that $\lambda \ge 0$ for all independent Stanley sequences. If κ is taken as small as possible, then a_{2^κ} is called the *repeat factor*; informally, it is the point at which the sequence begins its repetitive behavior. It is proven in [14] for any sufficiently large integer ρ, there is an independent Stanley sequence with repeat factor ρ.

Example 1.4. The sequence $S(0, 1, 7)$ is independent, with character $\lambda = 7$ and repeat factor $a_4 = 10$.

$$S(0, 1, 7) = 0, 1, 7, 8, 10, 11, 17, 18, 30, 31, 37, 38, 40, 41, 47, 48, 90, \dots.$$

Notice that the terms a_4, a_5, a_6, a_7 equal the terms a_0, a_1, a_2, a_3 increased by 10. Likewise, terms a_8 through a_{15} equal the terms a_0 through a_7 increased by 30; this illustrates the first condition of an independent sequence. Furthermore, the sequence approximately doubles when the index is a power of 2: between $a_3 = 8$ and $a_4 = 10$, between $a_7 = 18$ and $a_8 = 30$, and between $a_{15} = 48$ and $a_{16} = 90$. These jumps become increasingly evident as the index increases, since the character λ represents an additive correction; for instance, $a_{16} = 2 \cdot a_{15} - \lambda + 1$.

Definition 1.5. A Stanley sequence $S(A) = (a_n)$ is *regular* if there exist constants λ, σ and an independent Stanley sequence (a'_n), having character λ, such that, for large enough k and $0 \le i < 2^k$,

- $a_{2^k-\sigma+i} = a_{2^k-\sigma} + a'_i$,
- $a_{2^k-\sigma} = 2a_{2^k-\sigma-1} - \lambda + 1$.

The sequence (a'_n) is called the *core* of $S(A)$ and the constant σ is the *shift index*. We refer to λ as the *character* of (a_n) as well as of (a'_n).

Example 1.6. The sequence $S(0, 1, 4)$ is regular with core $(a'_n) = S(0)$, shift index $\sigma = 0$, and character $\lambda = 0$.

$$S(0, 1, 4) = 0, 1, 4, 5, 11, 12, 14, 15, 31, 32, 34, 35, 40, 41, 43, 44, 89, \dots.$$

Notice that the terms a_4, a_5, a_6, a_7 equal the terms a'_0, a'_1, a'_2, a'_3 of $S(0)$ increased by 11. Likewise, the terms a_8 through a_{15} equal the terms a'_0 through a'_7 increased by 31. As with an independent sequence, the sequence $S(0, 1, 4)$ has jumps when the index is a power of 2, for instance from 44 to 89. However, for some regular sequences, the shift index is nonzero, which means that the jumps are shifted away from powers of 2. For instance, by removing 11 from the sequence $S(0, 1, 4)$, while leaving all other terms unchanged, we obtain the sequence $S(0, 1, 4, 5, 12, 14, 15, 31)$, which is also regular and has shift index 1. In a sense, the term 11 is "unnecessary" in the Stanley sequence.