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Triangle-free graphs with the maximum number of cycles

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ABSTRACT

It is shown that for $n \geq 141$, among all triangle-free graphs on n vertices, the balanced complete bipartite graph $K_{\lceil n/2\rceil,\lfloor n/2\rfloor}$ is the unique triangle-free graph with the maximum number of cycles. Using modified Bessel functions, tight estimates are given for the number of cycles in $K_{\lceil n/2\rceil,\lfloor n/2\rfloor}$. Also, an upper bound for the number of Hamiltonian cycles in a triangle-free graph is given.

1. Introduction

All graphs in this paper are simple and undirected.

In a recent article [14], the maximum number of cycles in a triangle-free graph was considered. It was asked which triangle-free graphs contain the maximum number of cycles; this question arose from the study of path-finding algorithms [10]. The same authors posed the following conjecture:

Conjecture 1 (*Durocher–Gunderson–Li–Skala, 2014 [14]*). *For each $n \geq 4$, the balanced complete bipartite graph $K_{\lceil n/2\rceil,\lfloor n/2\rfloor}$ contains more cycles than any other n-vertex triangle-free graph.*

The authors [14] confirmed Conjecture 1 when $4 \leq n \leq 13$, and made progress toward this conjecture in general. For example, they showed the conjecture to be true when restricted to "nearly regular graphs", that is, for each positive integer k and sufficiently large n, $K_{\lceil n/2\rceil,\lfloor n/2\rfloor}$ has more cycles than any other triangle-free graph on n vertices whose minimum degree and maximum degree differ by at most k.

In Theorems 5.1 and 5.2, it is shown that Conjecture 1 holds true for $n \geq 141$. Theorem 3.4 gives a useful estimate for the number of cycles in $K_{\lceil n/2\rceil,\lfloor n/2\rfloor}$. In Lemma 4.3, an upper bound is given for the number of Hamiltonian cycles in a triangle-free graph.

Even though Conjecture 1 arose from a very specific problem in computing, it can be considered as a significant problem in two aspects of graph theory: counting cycles in graphs, and the structure of triangle-free graphs. In recent decades, bounds have been proved for the maximum number of cycles in various classes of graphs. Some of these classes include complete graphs [21]; planar graphs [4,5,11]; outerplanar graphs and series–parallel graphs [13]; graphs with large maximum degree without a specified odd cycle [8]; graphs with specified minimum degree [35]; graphs with a specified cyclomatic number or number of edges [2,23,15,19] (see also [25, Ch4, Ch10]); cubic graphs [3,12]; graphs with fixed girth [27];, k-connected

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graphs [24]; Hamiltonian graphs [28,31,35]; Hamiltonian graphs with a fixed number of edges [20]; 2-factors of the de Bruijn graph [17]; graphs with a cut-vertex [35]; complements of trees [22,29,36]; and random graphs [34]. In some cases, the structure of the extremal graphs are also found (see, *e.g.*, [8,28]).

In 1973, Erdős, Kleitman, and Rothschild [16] showed that for $r \geq 3$, as $n \to \infty$, the number of K_r-free graphs on n vertices is

$$2^{(1-\frac{1}{r-1}+o(1))\binom{n}{2}}.$$

As a consequence, the number of triangle-free graphs is very close to the number of bipartite graphs, and so almost all triangle-free graphs are bipartite. By Mantel's theorem [26], among graphs on n vertices, the triangle-free graph with the most number of edges is the balanced complete bipartite graph $K_{\lceil n/2\rceil,\lfloor n/2\rfloor}$.

Since $K_{\lceil n/2\rceil,\lfloor n/2\rfloor}$ is the triangle-free graph on n vertices with the most number of edges, and nearly all triangle-free graphs are bipartite, Conjecture 1 might seem reasonable, even though $K_{\lceil n/2\rceil,\lfloor n/2\rfloor}$ contains no odd cycles.

2. Notation and approximations used

A graph G is an ordered pair $G = (V, E) = (V(G), E(G))$, where V is a nonempty set and E is a set of unordered pairs from V. Elements of V are called vertices and elements of E are called edges. Under this definition, graphs are simple, that is, there are no loops nor multiple edges.

An edge $\{x, y\} \in E(G)$ is denoted by simply xy. The neighborhood of any vertex $v \in V(G)$ is $N_G(x) = \{y \in V(G) : xy \in E(G)\}$, and the degree of x is $\deg_G(x) = |N(x)|$. When it is clear what G is, subscripts are deleted, using only $N(x)$ and $\deg(x)$. The minimum degree of vertices in a graph G is denoted by $\delta(G)$, and the maximum degree is denoted $\Delta(G)$. If $Y \subset V(G)$, the subgraph of G induced by Y is denoted $G[Y]$.

A graph $G = (V, E)$ is called bipartite iff there is a partition $V = A \cup B$ so that $E \subset \{\{x, y\} : x \in A, y \in B\}$; if $E = \{\{x, y\} : x \in A, y \in B\}$, then G is called the complete bipartite graph on partite sets A and B, denoted $G = K_{|A|,|B|}$. The balanced complete bipartite graph on n vertices is $K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$. A cycle on m vertices is denoted C_m. The complement of a graph G is denoted $\overline{G}$. For any graph G, let $c(G)$ denote the number of cycles in G.

The number e is the base of the natural logarithm. The Stirling's approximation formula says that as $n \to \infty$,

$$n! = (1 + o(1))\sqrt{2\pi n}(n/e)^n. \tag{1}$$

In 1955, Robbins [30] proved that

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{1}{12n+1}} < n! < \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{1}{12n}}.$$

Slightly more convenient bounds (valid for all $n \geq 1$) are freely used in this paper (*e.g.*, in the proof of Theorem 3.4).

$$\sqrt{2\pi}\cdot\sqrt{n}\left(\frac{n}{e}\right)^n < n! \leq e\cdot\sqrt{n}\left(\frac{n}{e}\right)^n. \tag{2}$$

Two modified Bessel functions (see, *e.g.*, [1]) are used:

$$I_0(x) = \sum_{k=0}^{\infty}\frac{x^{2k}}{2^{2k}(i!)^2}; \tag{3}$$

$$I_1(x) = \sum_{k=0}^{\infty}\frac{x^{2k+1}}{2^{2k+1}i!(i+1)!}. \tag{4}$$

In particular, when $x = 2$ is used in either modified Bessel function, useful approximations are obtained:

$$2.27958 \leq \sum_{i=0}^{\infty}\frac{1}{(i!)^2} = I_0(2) \leq 2.279586; \tag{5}$$

$$1.5906 \leq \sum_{i=0}^{\infty}\frac{i}{(i!)^2} = \sum_{k=0}^{\infty}\frac{1}{k!(k+1)!} = I_1(2) \leq 1.59064. \tag{6}$$

3. Preliminaries

The following shows that among all bipartite graphs, the balanced one has the most cycles.

Lemma 3.1 ([*14*]). *For $n \geq 4$, among all bipartite graphs on n vertices, $K_{\lceil n/2\rceil,\lfloor n/2\rfloor}$ has the greatest number of cycles; that is, $K_{\lceil n/2\rceil,\lfloor n/2\rfloor}$ is the unique cycle-maximal bipartite graph on n vertices.*