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Ore-type degree conditions for disjoint path covers in simple graphs

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ABSTRACT

A *many-to-many k-disjoint path cover* of a graph joining two disjoint vertex subsets S and T of equal size k is a set of k vertex-disjoint paths between S and T that altogether cover every vertex of the graph. It is classified as *paired* if each source in S is required to be paired with a specific sink in T, or *unpaired* otherwise. In this paper, we develop Ore-type sufficient conditions for the existence of many-to-many k-disjoint path covers joining arbitrary vertex subsets S and T. Also, an Ore-type degree condition is established for the one-to-many k-disjoint path cover, a variant derived by allowing to share a single source. The bounds on the degree sum are all best possible.

1. Introduction

Let G be a simple undirected graph, whose vertex and edge sets are denoted by $V(G)$ and $E(G)$, respectively. The *order* of G is the number of vertices in G. If $(u, v) \in E(G)$, u is *adjacent* to v or u is a *neighbor* of v. A *path* from v_1 to v_m is a sequence of vertices, $(v_1, v_2, \ldots, v_m)$, such that v_j is adjacent to v_{j-1} for every $j \in \{2, \ldots, m\}$. A *disjoint path cover* (DPC for short) of G is a set of paths in G such that every vertex of G belongs to one and only one path.

Definition 1 (*Many-to-many k-disjoint path cover*)**.** Given two disjoint vertex subsets $S = \{s_1, \ldots, s_k\}$ and $T = \{t_1, \ldots, t_k\}$, each representing k *sources* and *sinks*, the *many-to-many k-disjoint path cover* of G is a disjoint path cover of size k, each of whose paths joins a pair of a source and a sink.

The disjoint path cover is *paired* if every source s_i must be joined with a specific sink t_i. On the other hand, it is *unpaired* if, for some permutation σ on $\{1, \ldots, k\}$, P_i is a path from s_i to $t_{\sigma(i)}$ for all $i \in \{1, \ldots, k\}$. The sources and sinks are referred to as *terminals*. By these definitions, a paired k-disjoint path cover is always an unpaired k-disjoint path cover. An example of the paired many-to-many DPC is shown in Fig. 1(a).

Definition 2. A graph G is *paired* (resp. *unpaired*) *many-to-many k-coverable* if $|V(G)| \geq 2k$ and there exists a paired (resp. unpaired) many-to-many k-DPC for any disjoint source set S and sink set T of size k each.

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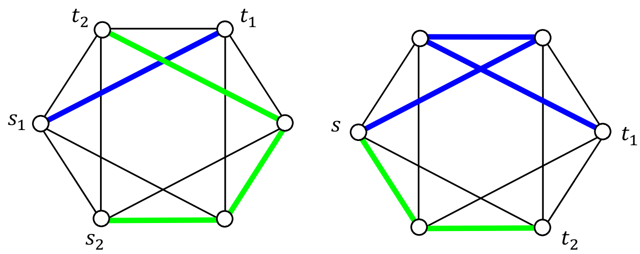

(a) Paired many-to-many 2-DPC. (b) One-to-many 2-DPC.

Fig. 1. Examples of disjoint path covers.

Two simpler variants of the many-to-many k-disjoint path cover can be derived by allowing to share a single source and/or a single sink. The first one is of *one-to-many* type with $S = \{s\}$ and $T = \{t_1, \ldots, t_k\}$, in which all paths start from the unique source s. The second one is of *one-to-one* type with $S = \{s\}$ and $T = \{t\}$, where all internally disjoint paths connect the unique source and sink. Refer to Fig. 1(b) for an example of the one-to-many DPC.

Definition 3. A graph G is *one-to-many* (resp. *one-to-one*) *k-coverable* if $|V(G)| \geq k+1$ and G has a one-to-many (resp. one-to-one) k-DPC for any disjoint source set $S = \{s\}$ and sink set $T = \{t_1, \ldots, t_k\}$ (resp. $T = \{t\}$).

The disjoint path cover problem is strongly related to the well-known Hamiltonian problem, which is a fundamental one in graph theory. Actually, a Hamiltonian path joining a pair of vertices in a graph forms a many-to-many, one-to-many, and one-to-one 1-disjoint path covers of the graph. A graph of order $n \geq 3$ is one-to-many 2-coverable if and only if it is Hamiltonian-connected. Moreover, a graph of order $n \geq 3$ is one-to-one 2-coverable if and only if it is Hamiltonian.

One of the core subjects in Hamiltonian graph theory is to develop sufficient conditions for a graph to have a Hamiltonian path/cycle (refer to [11] for a survey). The approaches taken to develop sufficient conditions usually involve degree conditions for providing enough edges to overcome any obstacle to the existence of a Hamiltonian path/cycle. Dirac [4] proved that a graph G of order $n \geq 3$ is Hamiltonian if $d_G(v) \geq n/2$ for every vertex v of G, where $d_G(v)$ denotes the degree of a vertex v in G. Ore [14,15] improved Dirac's condition as follows:

Theorem 1 (*Ore [14,15]*). *(a) A graph G of order $n \geq 3$ is Hamiltonian if $d_G(u) + d_G(v) \geq n$ for all distinct nonadjacent vertices u and v.*

(b) A graph G of order $n \geq 2$ is Hamiltonian-connected if $d_G(u) + d_G(v) \geq n+1$ for all distinct nonadjacent vertices u and v.

The close relationship between the disjoint path cover problem and the Hamiltonian problem motivates the study of developing degree conditions for a graph to have disjoint path covers. In this paper, we establish Ore-type conditions for the existence of k-disjoint path covers in a simple graph as follows:

1. A graph G of order $n \geq 2k$, where $k \geq 1$, is unpaired many-to-many k-coverable if $d_G(u) + d_G(v) \geq n + k$ for every pair of nonadjacent vertices u and v.
2. A graph G of order $n \geq 2k$, where $k \geq 2$, is paired many-to-many k-coverable if $d_G(u) + d_G(v) \geq n + 3k - 4$ for every pair of nonadjacent vertices u and v.
3. A graph G of order $n \geq k + 1$, where $k \geq 2$, is one-to-many k-coverable if $d_G(u) + d_G(v) \geq n + k - 1$ for every pair of nonadjacent vertices u and v.

Also, we show that all the above three bounds on the degree sum $d_G(u) + d_G(v)$ are the minimum possible. Note that the results for unpaired many-to-many and one-to-many disjoint path covers are generalizations of Ore's theorem, Theorem 1(b). For the one-to-one disjoint path cover problem, an Ore-type condition was derived by Lin et al. [12]: *A graph G of order $n \geq k+1$, where $k \geq 2$, is one-to-one k-coverable if $d_G(u)+d_G(v) \geq n+k-2$ for every pair of nonadjacent vertices u and v. Moreover, the bound on the degree sum is tight.* A one-to-one k-coverable graph is also known as a k^*-*connected* graph.

2. Related works and definitions

The disjoint path cover problem finds applications in many areas such as software testing, database design, and code optimization [1,13]. In addition, the problem is concerned with applications where full utilization of network nodes is important [18]. It has been studied with respect to various graphs such as hypercubes [3,5,6], recursive circulants [8,9,18, 19], hypercube-like graphs [7,10,18,19], cubes of connected graphs [16,17], k-ary n-cubes [20,21], and spanning connected graphs [12]. Unfortunately, it is NP-complete to determine whether or not there exists a many-to-many, one-to-many, or one-to-one k-DPC for a given pair of terminal sets in a general graph for any fixed $k \geq 1$ [18,19].

A *Hamiltonian path* of a graph G is a path that contains all the vertices of G. A graph is said to be *Hamiltonian-connected* if every pair of distinct vertices are joined by a Hamiltonian path. A *cycle* is a closed path of three or more vertices. A *Hamiltonian cycle* of G is a closed Hamiltonian path, i.e., a cycle that contains all the vertices of G. A graph is called *Hamiltonian* if it has