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Optimal unavoidable sets of types of 3-paths for planar graphs of given girth

S. Jendrol' [a], M. Maceková [a], M. Montassier [b], R. Soták [a,*]

[a] Institute of Mathematics, Faculty of Science, P. J. Šafárik University, Jesenná 5, 040 01 Košice, Slovakia

[b] University Montpellier 2, LIRMM, CNRS, UMR 5506, 161 Rue Ada, 34095 Montpellier Cedex 5, France

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ABSTRACT

In this paper we study unavoidable sets of types of 3-paths for families of planar graphs with minimum degree at least 2 and a given girth g. A 3-path of type (i, j, k) is a path uvw on three vertices u, v, and w such that the degree of u (resp. v, resp. w) is at most i (resp. j, resp. k). The elements i, j, k are called *parameters* of the type. The set S of types of paths is *unavoidable* for a family $\mathcal{F}$ of graphs if each graph G from $\mathcal{F}$ contains a path of the type from S. An unavoidable set S of types of paths is *optimal* for the family $\mathcal{F}$ if neither any type can be omitted from S, nor any parameter of any type from S can be decreased.

We prove that the set S_g (resp. S'_g) is an optimal set of types of 3-paths for the family of plane graphs having $\delta(G) \geq 2$ and girth $g(G) \geq g$ where

(i) $S_5 = \{(2, \infty, 2), (2, 3, 5), (2, 4, 3), (3, 3, 3)\}$,
(ii) $S_7 = \{(2, 3, 3), (2, 5, 2)\}$,
$S'_7 = \{(2, 2, 6), (2, 3, 3), (2, 4, 2)\}$,
(iii) $S_8 = \{(2, 2, 5), (2, 3, 2)\}$,
(iv) $S_{10} = \{(2, 4, 2)\}$,
$S'_{10} = \{(2, 2, 3), (2, 3, 2)\}$,
(v) $S_{11} = \{(2, 2, 3)\}$.

1. Introduction

In this paper we use a standard graph theory terminology according to the book [3]. However we recall here some notions.

Let G be a connected plane graph. We use $V(G)$, $E(G)$, $F(G)$, $\Delta(G)$, and $\delta(G)$ (or simply V, E, F, Δ, δ) to denote the vertex set, the edge set, the face set, the maximum degree, and the minimum degree of G, respectively. Faces of G are open 2-cells. The boundary of a face α is the boundary in the usual topological sense. It is a collection of all edges and vertices lying in the closure of a face α that can be organized into a closed walk in the graph G by traversing a simple closed curve just inside the face α. This closed walk is unique up to the choice of initial vertex and direction, and is called the *boundary walk* of the face α (see [11], p. 101).

The *degree* of a vertex v or a face α, that is the number of edges incident with v or the length of the boundary walk of α, is denoted by $\deg(v)$ or $\deg(\alpha)$, respectively. A *k-vertex* is a vertex v with $\deg(v) = k$. By k^+ or k^- we denote any integer not

* Corresponding author.
E-mail addresses: stanislav.jendrol@upjs.sk (S. Jendrol'), maria.macekova@student.upjs.sk (M. Maceková), mickael.montassier@lirmm.fr (M. Montassier), roman.sotak@upjs.sk (R. Soták).

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smaller or not greater than k, respectively. Hence, a k^+-*vertex* v (resp. k^+-*face* α) satisfies $\deg(v) \geq k$ (resp. $\deg(\alpha) \geq k$) and k^--*vertex* v (resp. k^--*face* α) satisfies $\deg(v) \leq k$ (resp. $\deg(\alpha) \leq k$). The *girth* $g(G)$ of G is the length of a shortest cycle in G. A k-*path* is a path on k vertices ($k \geq 1$). Let $w_k(G) = w_k$ be the minimum sum of degrees of vertices of a path on k vertices. A k-path on vertices $v_1, \dots, v_k$ is a *path of type* $(a_1, \dots, a_k)$ or an $(a_1, \dots, a_k)$-*path* if $\deg(v_i) \leq a_i$ for every $i \in \{1, \dots, k\}$. The elements $a_1, \dots, a_k$ are called *parameters* of the type. The set S of types of paths is *unavoidable* for a family $\mathcal{F}$ of graphs if each graph G from $\mathcal{F}$ contains a path of the type from S. An unavoidable set S of types of paths is *optimal* for the family $\mathcal{F}$ if neither the type can be omitted from S, nor any parameter of any type from S can be decreased.

In this paper we study unavoidable sets of types of 3-paths for families of planar graphs with minimum degree at least 2 and a given girth g. It is well known that every planar graph contains a vertex of degree at most 5. When the girth of a planar graph increases, one can guarantee the existence of a vertex with a smaller degree: every planar graph with girth at least 4 (resp. 6) contains a vertex of degree at most 3 (resp. at most 2). Moreover, if the girth is at least $5k + 1$ ($k \geq 1$), then a planar graph contains either a vertex of degree 1 or a k-path consisting of k vertices of degree 2 [20]. It is then natural to ask a similar question for larger structures, for example, for (i, j)-paths or (i, j, k)-paths. Concerning the existence of (i, j)-paths (such paths are also called *light edges*, see [17]) in a normal plane map[1], the effort of Lebesgue [19], Kotzig [18], Barnette [12] has flowed in the following theorem by Borodin [4]:

Theorem 1 (*Borodin [4]*)**.** *The set of types of* 2*-paths* $\{(3, 10), (4, 7), (5, 6)\}$ *is optimal for the family of normal plane maps.*

That result was then extended by Jendrol' and Maceková as follows:

Theorem 2 (*Jendrol' and Maceková [14]*)**.** *The set* P_g *is an optimal set of* 2*-paths for the family of plane graphs with minimum degree* $\delta(G) \geq 2$ *and girth* $g(G) \geq g$ *where*

(i) $P_5 = \{(2, 5), (3, 3)\}$,
(ii) $P_6 = \{(2, 5)\}$,
(iii) $P_7 = \{(2, 3)\}$,
(iv) $P_{11} = \{(2, 2)\}$.

In plane graphs with $\delta(G) \geq 2$ and $g(G) \leq 4$ there can exist a 2-path of the type $(2, r)$ for arbitrary $r \in \mathbb{N}$ (see the graph $K_{2,r}$).

Now, consider (i, j, k)-paths. The main motivation for our research comes from the following results:

Theorem 3 (*Franklin [10]*)**.** *Every normal plane map G such that* $\delta(G) = 5$ *contains a* $(6, 5, 6)$*-path.*

Theorem 4 (*Ando, Iwasaki, Kaneko [2]*)**.** *Every* 3*-polytope*[2] *satisfies* $w_3 \leq 21$*, which is tight.*

Theorem 5 (*Jendrol' [13]*)**.** *The set of types of* 3*-paths* $\{(10, 3, 10), (7, 4, 7), (6, 5, 6), (3, 4, 15), (3, 6, 11), (3, 8, 5), (3, 10, 3), (4, 4, 11), (4, 5, 7), (4, 7, 5)\}$ *is unavoidable for the family of* 3*-polytopes.*

Theorem 6 (*Borodin, Ivanova, Jensen, Kostochka, Yancey [6]*)**.** *The set of types of* 3*-paths* $\{(3, 3, \infty), (3, 4, 11), (3, 7, 5), (3, 10, 4), (3, 15, 3), (4, 4, 9), (6, 4, 8), (6, 5, 6), (7, 4, 7)\}$ *is optimal for the family of normal plane maps.*

Theorem 7 (*Borodin, Ivanova, Kostochka [7]*)**.** *The set of types of* 3*-paths* $S = \{(3, 3, \infty), (3, 4, 11), (3, 8, 5), (3, 10, 4), (3, 15, 3), (4, 4, 9), (4, 7, 4), (5, 5, 7), (6, 4, 7), (6, 5, 6)\}$ *is optimal for the family of normal plane maps.*

Theorems 6 and 7 provide two uncomparable optimal sets of types of 3-paths for the family of normal plane maps. In [7] there is formulated an interesting problem of determining the exact list of distinct optimal sets of types of 3-paths for given families of plane graphs. Borodin and Ivanova in [5] give seven mutually uncomparable optimal sets of types of 3-paths for triangle-free normal plane maps.

Theorem 8 (*Borodin, Ivanova [5]*)**.** *There exist precisely seven optimal sets of types of* 3*-paths for triangle-free normal plane maps:*

(i) $\{(5, 3, 6), (4, 3, 7)\}$,
(ii) $\{(3, 5, 3), (3, 4, 4)\}$,
(iii) $\{(5, 3, 6), (3, 4, 3)\}$,
(iv) $\{(3, 5, 3), (4, 3, 4)\}$,
(v) $\{(5, 3, 7)\}$,
(vi) $\{(3, 5, 4)\}$,
(vii) $\{(5, 4, 6)\}$.

[1] A *normal plane map* is a plane graph in which loops and multiple edges are allowed, but the degree of each vertex and face is at least three.
[2] 3-*polytopes* are precisely 3-connected planar graphs (Steinitz's theorem).