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Note

Triangle-free oriented graphs and the traceability conjecture

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ABSTRACT

For $k \geq 2$, an oriented graph D of order at least k, is said to be k-traceable if any subset of k vertices of D induces a traceable oriented graph. The traceability conjecture asserts that every k-traceable oriented graph of order $n \geq 2k-1$ is traceable. In this paper we prove that the traceability conjecture is true for triangle-free oriented graphs of order $n = 2k-1$ or $n \geq 3k-7$. In a second section, we prove that the traceability conjecture is true for oriented graphs of girth at least 5.

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1. Notation and background

For a digraph D, $V(D)$ is the set of the vertices of D, and $\mathcal{A}(D)$ is the set of the arcs of D. The number of the vertices of D is the *order* of D and the number of arcs is the *size* of D and is denoted by $n(D)$ and $m(D)$ respectively. Where no confusion arises, we will suppress the D. We say that a vertex y is an *out-neighbor* of a vertex x (*in-neighbor* of x) if (x, y) (resp. (y, x)) is an arc of D. $N_D^+(x)$ is the set of the out-neighbors of x, and $N_D^-(x)$ is the set of the in-neighbors of x. The cardinality of $N_D^+(x)$ is the *out-degree* $d_D^+(x)$ of x and the cardinality of $N_D^-(x)$ is the *in-degree* $d_D^-(x)$ of x. A vertex y is *adjacent* to a vertex x of D, if y is either an out-neighbor of x or an in-neighbor of x. For a set X of vertices of D, we denote by $D[X]$ the sub-digraph of D induced by X, that is the digraph whose vertex set is X and whose arcs are the arcs of D with both vertices in X. The set of the vertices of D distinct from x and non-adjacent to x is denoted $N^0(x)$. If X and Y are vertex disjoint subsets of $V(D)$, we say that X *weakly dominates* Y, if there are no arcs from any vertex of Y to any vertex of X. By paths or cycles, we always mean directed paths or directed cycles. A cycle of length 3 is a triangle, and a triangle-free digraph is a digraph which does not contain triangles. If $P = x_1 \ldots x_p$ and $Q = y_1 \ldots y_q$ are two vertex-disjoint paths of the digraph D, and if (x_p, y_1) is an arc of D, then PQ is the path $x_1 \ldots x_p y_1 \ldots y_q$. If D contains cycles, the *girth* $g(D)$ of D is the minimum order of the cycles of D. A *hamiltonian cycle* of a digraph D is a cycle of D containing all the vertices of D (and then D is said to be hamiltonian). A *hamiltonian path* of D is a path containing all the vertices of D (and then D is said to be *traceable*). For $k \geq 2$, a *k-traceable digraph* is a digraph of order at least k, such that for any set X of k vertices of D, the sub-digraph induced by X, is traceable.

An *oriented graph* is a digraph D such that for any distinct vertices x, y of D, at most one of the ordered pairs (x, y) and (y, x) is an arc of D. A *tournament* is an oriented graph T such that for any distinct vertices x, y of D, exactly one of the ordered pairs (x, y) and (y, x) is an arc of D. It is well known that every tournament is traceable and that every strong tournament is hamiltonian. For undefined notation and definition we refer the reader to [1].

The traceability conjecture of Van Aardt, Dunbar, Frick, Nielsen and Oellermann (see [5]), asserts that for $k \geq 2$, every k-traceable oriented graph of order at least $2k-1$ is traceable. In the same paper, the authors proved that the conjecture is

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true for $2 \le k \le 5$. Frick and Katrenič [2] proved the case $k = 6$. Recently Van Aardt et al. proved that the conjecture is true for $k = 7$ and $k = 8$ (see [3]) The results of [5,2] and [3] give:

Theorem 1.1. **(a)** *For $2 \le k \le 6$, every k-traceable oriented graph is traceable.*
(b) *Every 7-traceable oriented graph of order at least 9 is traceable.*
(c) *Every 8-traceable oriented graph of order at least 14 is traceable.*

It was proved in [4] that for $k \ge 9$, every k-traceable oriented graph of order at least $2k^2 - 20k + 59$ is traceable. In this paper we improve this result to a linear bound for triangle-free oriented graphs. Namely, we prove:

Theorem 1.2. *For $k \ge 2$, every k-traceable triangle-free oriented graph of order $n = 2k - 1$ is traceable.*

Theorem 1.3. *For $k \ge 2$, every k-traceable triangle-free oriented graph D of order $n \ge 3k - 7$ is traceable.*

Theorem 1.4. *For $k \ge 2$, every k-traceable oriented graph of girth at least 5 and of order $n \ge 2k - 1$ is traceable.*

Thus, Theorem 1.4 implies that the traceability conjecture is true for k-traceable oriented graphs of girth at least 5. We will use two intermediate results (see [5] for the first result, and [4] for the second result):

Lemma 1.5. *For $k \ge 2$, let D be a k-traceable oriented graph of order n. Then for every vertex x of D, $d^+(x) + d^-(x) \ge n - k + 1$.*

Lemma 1.6. *For $k \ge 2$, let D be a k-traceable oriented graph of order n. Then for every pair of non-adjacent vertices x and y, $|N^+(x) \cup N^+(y)| \ge n - k + 1$ and $|N^-(x) \cup N^-(y)| \ge n - k + 1$.*

Clearly this last lemma implies that for non-adjacent vertices x and y, it holds $d^+(x) + d^+(y) \ge n - k + 1$ and $d^-(x) + d^-(y) \ge n - k + 1$.

2. Proof of Theorem 1.2

We proceed by induction on k. The assertion is true for $k \le 8$ (see [5,2] and [3]). Consider an integer $k > 8$ and suppose that for every $j < k$, every triangle-free j-traceable oriented graph of order $2j - 1$ is traceable. Now let D be a triangle-free k-traceable oriented graph D of order $n = 2k - 1$. Suppose, to the contrary that D is not traceable.

First we define the sets $A = \{x \in V(D) : d^+(x) \ge k\}$ and $B = \{x \in V(D) : d^-(x) \ge k\}$, and let a and b the respective cardinalities of A and B. Without loss of generality, we may suppose that $a \ge b$. We claim the following.

Claim 2.1. **(i)** *$A \cup B = V(D)$ and $A \cap B = \emptyset$;*
(ii) *$A \ne \emptyset$ and $B \ne \emptyset$;*
(iii) *A weakly dominates B;*
(iv) *$D[B]$ is a transitive tournament;*
(v) *$b \ge 2$.*

Proof. **(i)** Let x be a vertex of D. Without loss of generality, we may suppose that $d^+(x) \ge d^-(x)$. Suppose that $d^+(x) \le k - 1$. Then $|N^-(x)| \le k - 1$ and $|V(D) \setminus N^+(x)| \ge n - k + 1$, hence $|V(D) \setminus N^+(x)| \ge k$. This implies that there exists a set S of k vertices, contained in $V(D) \setminus N^+(x)$, and containing $N^-(x)$, x and possibly vertices non-adjacent with x. Consequently, there exists a hamiltonian path P' of $D[S]$, and necessarily the ending vertex of P' is x. We have $|S'| = k$, where $S' = (V(D) \setminus S) \cup \{x\}$. Then, there exists a hamiltonian path P'' of $D[S']$ and necessarily the starting point of this path is x. Let $P'' = xx_1 \dots x_{k-1}$ be this path. Then $P = P'x_1 \dots x_{k-1}$ is a hamiltonian path of D, which is impossible. So, it holds $d^+(x) \ge k$, which means $x \in A$. It follows $A \cup B = V(D)$. The sets A and B are disjoint (for otherwise we would have a vertex x with $d^+(x) + d^-(x) \ge 2k > n - 1$). This proves (i). Note that $a > b$ (because n is odd).

(ii) In the opposite case, we would have $m(D) \ge (2k - 1)k$ and since the maximum size of D is $\frac{(2k-1)(2k-2)}{2} = (2k - 1)(k - 1)$, we would get $(2k - 1)(k - 1) \ge (2k - 1)k$ which is false. This proves (ii).

(iii) Suppose the opposite and then let (x, y) be an arc of D with $x \in B$ and $y \in A$. Since $d^+(y) \ge k$ and $d^-(x) \ge k$, it follows $d^+(y) + d^-(x) \ge 2k > n - 2$. Consequently there exists a vertex $z \in N^+(y) \cap N^-(x)$. But then $xyzx$ is a triangle of D which is not possible. So, the assertion is proved.

(iv) Suppose to the contrary that $D[B]$ is not a tournament. Then there exists two non-adjacent vertices x and y of B. Since $a \ge k$, we can consider a set S' of $k - 2$ vertices of A, and then $S = S' \cup \{x, y\}$ is a set of k vertices. Consequently, $D[S]$ admits a hamiltonian path P. Since there are no arcs from $\{x, y\}$ to S', it is easy two see that the two last vertices of P are x and y. Then x and y are adjacent a contradiction. So, $D[B]$ is a tournament, and since D does not contain triangles, it is a transitive tournament.

(v) It is easy to see that $b = 1$ is not possible. Indeed, there exists a vertex u of out-degree at most $\frac{a-1}{2}$ in $D[A]$, and then we have $\frac{a-1}{2} + 1 \ge k$, hence $a \ge 2k - 1$, which is not possible. It follows $b \ge 2$ and (v) is proved. ■

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