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The existence spectrum for large sets of pure Hybrid triple systems☆

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ABSTRACT

An *LPHTS*(v) is a collection of $4(v-2)$ disjoint pure Hybrid triple systems on the same set of v elements. In Fan (2010), it is showed that there exists an *LPHTS*(v) for $v \equiv 0, 4 \bmod 6$. In this paper, we establish the existence of an *LPHTS*(v) for $v \equiv 1, 3 \bmod 6$, $v > 3$. Finally, the spectrum for *LPHTSs* is completely determined.

1. Introduction

Let X be a finite set. In what follows, an ordered pair of X is always an ordered pair (x, y), where $x \neq y \in X$. A *cyclic triple* $\langle x, y, z\rangle$ on X is a set of three ordered pairs (x, y), (y, z) and (z, x) of X. A *transitive triple* (x, y, z) on X is a set of three ordered pairs (x, y), (y, z) and (x, z) of X.

An *oriented triple system* of order v is a pair $(X, \mathcal{B})$ where X is a v-set and $\mathcal{B}$ is a collection of oriented triples on X, called *blocks*, such that each ordered pair of X belongs to exactly one block of $\mathcal{B}$. If $\mathcal{B}$ contains both cyclic triples and transitive triples, then $(X, \mathcal{B})$ is called a *Hybrid triple system* and denoted by *HTS*(v) which is firstly defined by *Colbourn, Pulleyblank and Rosa* in [1]. If the triples in $\mathcal{B}$ are all cyclic (or transitive), then $(X, \mathcal{B})$ is called a *Mendelsohn* (or *directed*) *triple system* and denoted by *MTS*(v) (or *DTS*(v)). A *holey Hybrid* (or *Mendelsohn*, or *directed*) *triple system HTS*(v, w) (or *MTS*(v, w), or *DTS*(v, w)) is a trio $(X, Y, \mathcal{B})$ where X is a v-set, Y is its w-subset (called the *hole*), $\mathcal{B}$ is a collection of both cyclic triples and transitive triples (or cyclic triples or transitive triples) of X such that each ordered pair of X but those on Y is exactly contained in one block of $\mathcal{B}$ and no ordered pair of Y is contained in any block of $\mathcal{B}$.

An *HTS*(v) (or *HTS*(v, w)) is called *pure* and denoted by *PHTS*(v) (or *PHTS*(v, w)) if at most one triple in the set $\{(x, y, z), (y, z, x), (z, x, y), (z, y, x), (y, x, z), (x, z, y), \langle x, y, z\rangle, \langle z, y, x\rangle\}$ is contained in $\mathcal{B}$. An *MTS*(v) (or *MTS*(v, w)) is called *pure* and denoted by *PMTS*(v) (or *PMTS*(v, w)) if $\langle x, y, z\rangle \in \mathcal{B}$ implies $\langle z, y, x\rangle \notin \mathcal{B}$. A *DTS*$(v)$ (or *DTS*(v, w)) is called *pure* and denoted by *PDTS*(v) (or *PDTS*(v, w)) if $(x, y, z) \in \mathcal{B}$ implies $(z, y, x) \notin \mathcal{B}$.

A *large set* of *PHTS*(v)*s* (or *PMTS*(v)*s* or *PDTS*(v)*s*), denoted by *LPHTS*(v) (or *LPMTS*(v) or *LPDTS*(v)), is a collection $\{(Y, \mathcal{A}_i)\}_i$ where Y is a v-set and each $(Y, \mathcal{A}_i)$ is a *PHTS*(v) (or *PMTS*(v) or *PDTS*(v)) and these $\mathcal{A}_i$s form a partition of all cyclic triples and transitive triples (or cyclic triples or transitive triples) on Y.

About the existence of the large sets of pure oriented triple system, there are the following known results.

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Lemma 1.1.
(1) [10] *There exists an LPMTS*(v) *if and only if* $v \equiv 0, 1 \bmod 3$, $v \geq 4$ *and* $v \neq 6, 7$.
(2) [9] *There exists an LPDTS*(v) *if and only if* $v \equiv 0, 1 \bmod 3$ *and* $v \geq 4$.

In this paper, we focus on the existence of *LPHTSs*. Some preliminary researches on *LPHTSs* have been done. We list some known results as follows.

Lemma 1.2 ([2]). *For odd* v, *if there exists an LPHTS*$(v+1)$, *then there exists an LPHTS*$(3v+1)$.

Lemma 1.3 ([3]).
(1) *[Theorem 4.2.1] For* $u \equiv 1, 5 \mod 6$ *and* $v > 3$, *if there exists an LPHTS*$(v+2)$, *then there exists an LPHTS*$(uv+2)$;
(2) *[Theorem 4.3.4] There exists an LPHTS*$(2u+2)$ *for* $u \equiv 1, 5 \mod 6$;
(3) *[Theorem 4.4.4] There exists an LPHTS*(2^n+2) *for* $n \geq 1$ *and* $n \neq 4$.

Fan actually has given the existence of *LPHTS*(v) for $v \equiv 0, 4 \bmod 6$ in [3]. If $v \equiv 0, 4 \bmod 6$ and $v > 0$, we can rewrite it as $v = 2^n u + 2$, where $u \equiv 1, 5 \bmod 6$ and $n \geq 1$. If $n = 1$, then there exists an *LPHTS*$(2u+2)$ by Lemma 1.3(2). If $n > 1$, since there exists an *LPHTS*(2^n+2) by Lemma 1.3(3) and Lemma 3.1(1), there exists an *LPHTS*$(2^n u+2)$ by Lemma 1.3(1). Thus, we have the following conclusion.

Lemma 1.4. *There exists an LPHTS*(v) *for* $v \equiv 0, 4 \bmod 6$.

Lemma 1.5. *There exists an LPHTS*(v) *only if* $v \equiv 0, 1 \bmod 3$ *and* $v > 3$.

Proof. Let $(Y, \mathcal{A})$ be a *PHTS*(v) where Y is a v-set and $\mathcal{A}$ is the block set, then $\mathcal{A}$ contains $\frac{v(v-1)}{3}$ blocks. Thus, $3|v(v-1)$. Evidently, there exists no *PHTS*(3). So, we get the necessary condition. ■

By Lemmas 1.4 and 1.5, in order to obtain the existence spectrum of *LPHTSs*, we only need to discuss the existence of *LPHTS*(v) for $v \equiv 1, 3 \bmod 6$. In Section 2, we display some recursive constructions. In Sections 3–5, we discuss the existence of an *LPHTS*(v) for $v = 6k+3$, $12k+7$, $12k+1$ respectively. Especially, we get the existence of *LPHTS*$(12k+7)$ by using a *PPHGDD* to construct a *PPHCS*. Finally, the existence spectrum for *LPHTSs* is determined.

2. Recursive construction

In this section, we described some definitions and present some useful constructions. First, we introduce four definitions about *HCS*, *PHGDD*, *t-PPHCS* and *LPHTS*(v, w), where the definition of *HCS* is a simple modification to a candelabra t-system in [6].

A Hybrid candelabra system (*HCS* for short) of order v is a quadruple $(X, S, \mathcal{G}, \mathcal{A})$ that satisfies the following properties:

(1) X is a set of v elements (called the *points*) and S is a s-subset (called the *stem*) of X;

(2) $\mathcal{G}$ is a collection of subsets (called the *groups*) of $X \backslash S$ which partition $X \backslash S$;

(3) $\mathcal{A}$ is a family of cyclic triples and transitive triples (called the *blocks*) on X such that every cyclic triple $\langle a, b, c\rangle$ or transitive triple (a, b, c) on X with $|\{a, b, c\} \cap (S \cup G)| < 3$ for any $G \in \mathcal{G}$ is contained in exactly one block, and for each $G \in \mathcal{G}$ no cyclic triple or transitive triple on $S \cup G$ is contained in any block.

The list $(\{|G| : G \in \mathcal{G}\} : s)$ is called *type* of the *HCS*. We also use the exponential notation to denote the type of $\mathcal{G}$ and separate the stem size by a colon.

For positive integers n_i and g_i, $1 \leq i \leq r$, an *HGDD*$(g_1^{n_1} \cdots g_r^{n_r})$ (or *MGDD*$(g_1^{n_1} \cdots g_r^{n_r})$) is a trio $(X, \mathcal{G}, \mathcal{A})$ satisfying the following conditions: (1) X is a set containing $\sum_{i=1}^{r} n_i g_i$ points; (2) $\mathcal{G}$ is a partition of X, which consists of n_i subsets (called the *groups*) of cardinality g_i; (3) $\mathcal{A}$ is a family of some cyclic triples and transitive triples (or some cyclic triples) of X (called the *blocks*) such that $|A \cap G| \leq 1$, $\forall A \in \mathcal{A}$, $G \in \mathcal{G}$ and each ordered pair on X from distinct (or the same) groups is contained in exactly one (or no) block.

An *HGDD*$(g_1^{n_1} \cdots g_r^{n_r})(X, \mathcal{G}, \mathcal{A})$ is called *pure* and denoted by *PHGDD*$(g_1^{n_1} \cdots g_r^{n_r})$ if at most one triple in the set $\{(x, y, z), (y, z, x), (z, x, y), (z, y, x), (y, x, z), (x, z, y), \langle x, y, z\rangle, \langle z, y, x\rangle\}$ is contained in $\mathcal{A}$. An *MGDD*$(g_1^{n_1} \cdots g_r^{n_r})(X, \mathcal{G}, \mathcal{A})$ is called *pure* and denoted by *PMGDD*$(g_1^{n_1} \cdots g_r^{n_r})$ if $\langle x, y, z\rangle \in \mathcal{A}$ implies $\langle z, y, x\rangle \notin \mathcal{A}$.

An special *PMGDD* can generate four disjoint *PHGDDs*. We have discussed the relation between them in [4], see Lemma 2.1 where the block-incident graph $G(\mathcal{B})$ for a *PMGDD*$(g_1^{\alpha_1} g_2^{\alpha_2} \cdots g_r^{\alpha_r})$ $(X, \mathcal{B})$ is defined as follows: the vertex set is $\mathcal{B}$, the vertices B and B' are jointed by an edge if and only if $|B \cap B'| = 2$.

Lemma 2.1 ([4, Corollary 2.2]). *Let* G *be the block-incident graph of a PMGDD*$(g_1^{\alpha_1} g_2^{\alpha_2} \cdots g_r^{\alpha_r})$ *and the block set is* $\mathcal{B}$. *If there exists a* 2*-factor of* G *consisting of some disjoint circuits with even length no less than* 4 *and* $|\mathcal{B}| > 6$, *then there exist four pairwise disjoint PHGDD*$(g_1^{\alpha_1} g_2^{\alpha_2} \cdots g_r^{\alpha_r})$*s*.