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Convexity in partial cubes: The hull number

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ABSTRACT

We prove that the combinatorial optimization problem of determining the hull number of a partial cube is NP-complete. This makes partial cubes the minimal graph class for which NP-completeness of this problem is known and improves earlier results in the literature.

On the other hand we provide a polynomial-time algorithm to determine the hull number of planar partial cube quadrangulations.

Instances of the hull number problem for partial cubes described include poset dimension and hitting sets for interiors of curves in the plane.

To obtain the above results, we investigate convexity in partial cubes and obtain a new characterization of these graphs in terms of their lattice of convex subgraphs. This refines a theorem of Handa. Furthermore we provide a topological representation theorem for planar partial cubes, generalizing a result of Fukuda and Handa about tope graphs of rank 3 oriented matroids.

1. Introduction

The objective of this paper is the study of convexity and particularly of the hull number problem on different classes of partial cubes. Our contribution is twofold. First, we establish that the hull number problem is NP-complete for partial cubes, second, we emphasize reformulations of the hull number problem for certain classes of partial cubes leading to interesting problems in geometry, poset theory and plane topology. In particular, we provide a polynomial time algorithm for the class of planar partial cube quadrangulations.

Denote by Q^d the hypercube graph of dimension d. A graph G is called a *partial cube* if there is an injective mapping $\phi : V(G) \to V(Q^d)$ such that $d_G(v, w) = d_{Q^d}(\phi(v), \phi(w))$ for all $v, w \in V(G)$, where, d_G and d_{Q^d} denote the graph distance in G and Q^d, respectively. This is, for each pair of vertices of $\phi(G)$, at least one shortest path in Q^d belongs to $\phi(G)$. In other words $\phi(G)$, seen as a subgraph of Q^d, is an *isometric embedding* of G in Q^d. One often does not distinguish between G and $\phi(G)$ and just says that G is an *isometric subgraph* of Q^d.

Partial cubes were introduced by Graham and Pollak in [23] in the study of interconnection networks and continue to find strong applications; they form for instance the central graph class in media theory (see the recent book [17]) and frequently appear in chemical graph theory e.g. [16]. Furthermore, partial cubes "present one of the central and most studied classes of graphs in all of the metric graph theory", citing [30].

Partial cubes form a generalization of several important graph classes, thus have also many applications in different fields of mathematics. This article discusses some examples of such families of graphs including Hasse diagrams of upper locally distributive lattices or equivalently antimatroids [19] (Section 6), region graphs of halfspaces and hyperplanes (Section 3),

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and tope graphs of oriented matroids [8] (Section 5). These families contain many graphs defined on sets of combinatorial objects: flip-graphs of strongly connected and acyclic orientations of digraphs [9], linear extension graphs of posets [33] (Section 4), integer tensions of digraphs [19], configurations of chip-firing games [19], to name a few.

Convexity for graphs is the natural counterpart of Euclidean convexity and is defined as follows; a subgraph G' of G is said to be *convex* if all shortest paths in G between vertices of G' actually belong to G'. The *convex hull* of a subset V' of vertices – denoted $\mathrm{conv}(V')$ – is defined as the smallest convex subgraph containing V'. Since the intersection of convex subgraphs is clearly convex, the convex hull of V' is the intersection of all the convex subgraphs that contain V'.

A subset of vertices V' of G is a *hull set* if and only if $\mathrm{conv}(V') = G$. The *hull number* or *geodesic hull number* of G, denoted by $hn(G)$, is the size of a smallest hull set. It was introduced in [18], and since then has been the object of numerous papers. Most of the results on the hull number are about bounds for specific graph classes, see e.g. [11,27,6,5,15,10]. Only recently, in [14] the focus was set on computational aspects of the hull number and it was proved that determining the hull number of a graph is NP-complete. In particular, computing the convex hull of a given set of vertices was shown to be polynomial time solvable. The NP-completeness result was later strengthened to bipartite graphs in [1]. On the other hand, polynomial-time algorithms have been obtained for unit-interval graphs, cographs and split graphs [14], cactus graphs and P_4-sparse graphs [1], distance hereditary graphs and chordal graphs [29], and P_5-free triangle-free graphs in [2]. Moreover, in [2], a fixed parameter tractable algorithm to compute the hull number of any graph was obtained. Here, the parameter is the size of a vertex cover.

Let us end this introduction with an overview of the results and the organization of this paper. Section 2 is devoted to properties of convexity in partial cubes and besides providing tools for the other sections, its purpose is to convince the reader that convex subgraphs of partial cubes behave nicely. A characterization of partial cubes in terms of their convex subgraphs is given. In particular, convex subgraphs of partial cubes behave somewhat like polytopes in Euclidean space. Namely, they satisfy an analogue of the Representation Theorem of Polytopes [36].

In Section 3 the problem of determining the hull number of a partial cube is proved to be NP-complete, improving earlier results of [14] and [1]. Our proof indeed implies an even stronger result namely that determining the hull number of a region graph of an arrangement of halfspaces and hyperplanes in Euclidean space is NP-complete.

In Section 4 the relation between the hull number problem for linear extension graphs and the dimension problem of posets is discussed. We present a quasi-polynomial-time algorithm to compute the dimension of a poset given its linear extension graph and conjecture that the problem is polynomial-time solvable.

Section 5 is devoted to planar partial cubes. We provide a new characterization of this graph class, which is a topological representation theorem generalizing work of Fukuda and Handa on rank 3 oriented matroids [21]. This characterization is then used to obtain a polynomial-time algorithm that computes the hull number of planar partial cube quadrangulations. We conjecture the problem to be polynomial time solvable for general planar partial cubes.

In Section 6 we study the lattice of convex subgraphs of a graph. First, we prove that given this lattice the hull-number of a graph can be determined in quasi-polynomial time and conjecture that the problem is indeed polynomial time solvable. We then prove that for any vertex v in a partial cube G, the set of convex subgraphs of G containing v ordered by inclusion forms an upper locally distributive lattice. This leads to a new characterization of partial cubes, strengthening a theorem of Handa [25].

We conclude the paper by giving the most interesting open questions in Section 7.

2. Partial cubes and cut-partitions

All graphs studied in this article are connected, simple and undirected. Given a connected graph G a *cut* $C \subseteq E$ is a set of edges whose removal partitions G into exactly two connected components. These components are called its *sides* and are denoted by C^+ and C^-. For $V' \subset V$, a cut C *separates* V' if both $C^+ \cap V'$ and $C^- \cap V'$ are not empty. A *cut-partition* of G is a set $\mathcal{C}$ of cuts partitioning E. For a cut $C \in \mathcal{C}$ and $V' \subseteq V$ define $C(V')$ as G if C separates V' and as the side of C containing V', otherwise.

Observation 1. *A graph G is bipartite if and only if G has a cut-partition.*

The equivalence classes of the *Djoković–Winkler relation* of a partial cube [13,34] can be interpreted as the cuts of a cut-partition. The following characterization of partial cubes is a reformulation of some properties of the Djoković–Winkler equivalence classes as well as some results from [7,3]. We provide a simple self-contained proof.

Theorem 2. *A connected graph G is a partial cube if and only if G admits a cut-partition $\mathcal{C}$ satisfying one of the following equivalent conditions:*

(i) *there is a shortest path between any pair of vertices using no $C \in \mathcal{C}$ twice*
(ii) *no shortest path in G uses any $C \in \mathcal{C}$ twice*
(iii) *for all $V' \subseteq V$: $\mathrm{conv}(V') = \bigcap_{C \in \mathcal{C}} C(V')$*
(iv) *for all $v, w \in V$: $\mathrm{conv}(\{v, w\}) = \bigcap_{C \in \mathcal{C}} C(\{v, w\})$.*

We call such a cut-partition a convex cut-partition.