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Planar graphs without cycles of length 4 or 5 are (2, 0, 0)-colorable[☆]

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ABSTRACT

Let $d_1, d_2, \ldots, d_k$ be k nonnegative integers. A graph $G = (V, E)$ is $(d_1, d_2, \ldots, d_k)$-colorable, if the vertex set V of G can be partitioned into subsets $V_1, V_2, \ldots, V_k$ such that the subgraph $G[V_i]$ induced by V_i has maximum degree at most d_i for $i = 1, 2, \ldots, k$. Steinberg conjectured that planar graphs without cycles of length 4 or 5 are (0, 0, 0)-colorable. Hill et al. showed that every planar graph without cycles of length 4 or 5 is (3, 0, 0)-colorable. In this paper, we show that planar graphs without cycles of length 4 or 5 are (2, 0, 0)-colorable. For further study in this direction, some problems and conjectures are presented.

1. Introduction

All graphs considered here are finite, simple and undirected. The notation and terminology used but undefined in this paper can be found in the book by Bondy and Murty [4].

Let $G = (V, E)$ be a graph with the sets of vertices and edges V and E, respectively. A *k-coloring* of G is a mapping $\phi : V \longrightarrow \{1, 2, \ldots, k\}$ such that $\phi(u) \neq \phi(v)$ whenever $uv \in E$. G is said to be *k-colorable* if G admits a k-coloring. It is well-known that (4CT) every planar graph is 4-colorable [2,3]; and (3CT) every triangle-free planar graph is 3-colorable [10]. For 3-colorability of planar graphs, an attractive central conjecture, proposed by Steinberg [4,17], says that every planar graph without cycles of length 4 or 5 is 3-colorable.

As a relaxation of the Steinberg's conjecture, Erdös [17] suggested to determine the minimum number k such that every planar graph without cycles of length from 4 to k is 3-colorable. Abbott and Zhou [1] first showed that $k \leq 11$. This bound was later on improved to 9 by Borodin [5] and, independently, by Sanders and Zhao [16]; to 7 by Borodin et al. [6].

Another natural approach to attack the Steinberg's conjecture seems to study the *improper colorings* (or *defective colorings* in some earlier papers) of the planar graphs without cycles of length 4 or 5. Let $d_1, d_2, \ldots, d_k$ be k nonnegative integers. A $(d_1, d_2, \ldots, d_k)$-*coloring* of a graph $G = (V, E)$ is a mapping $\phi : V \longrightarrow \{1, \ldots, k\}$ such that the subgraph $G[V_i]$ induced by V_i has maximum degree at most d_i, where $V_i = \{v \in V | \phi(v) = i\}$. G is $(d_1, d_2, \ldots, d_k)$-*colorable* if it admits a $(d_1, d_2, \ldots, d_k)$-coloring. Note that if G is $(d_1, d_2, \ldots, d_k)$-colorable, then it is $(d'_1, d'_2, \ldots, d'_k)$-colorable, whenever $0 \leq d_i \leq d'_i$ for all $i = 1, 2, \ldots, k$.

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In terms of the general notion above, the four color theorem (4CT) may be restated as: every planar graph is (0, 0, 0, 0)-colorable, and the Steinberg's conjecture may be restated as: every planar graph without cycles of length 4 or 5 is (0, 0, 0)-colorable. It is known that every planar graph is (2, 2, 2)-colorable [9]. Let $\mathscr{F}$ denote the family of planar graphs without cycles of length 4 or 5. Motivated by the Steinberg's conjecture, Lih et al. [14] showed that every graph in $\mathscr{F}$ is list (1, 1, 1)-colorable, hence, (1, 1, 1)-colorable; Chang et al. [8] showed that every graph in $\mathscr{F}$ is (4, 0, 0)- and (2, 1, 0)-colorable. Recently, Hill et al. [11] showed that every graph in $\mathscr{F}$ is (3, 0, 0)-colorable. Moreover, Hill and Yu [12], and independently, Xu, Miao and Wang [25], showed that every graph in $\mathscr{F}$ is (1, 1, 0)-colorable. In this paper, we prove the following result.

Theorem 1. *Every graph in $\mathscr{F}$ is* (2, 0, 0)*-colorable.*

For other related papers on improper colorability of planar graphs with some forbidden cycles, we refer the readers to [7–13,15,18–23,26].

The rest of this section is devoted to some definitions. A *plane* graph is an embedding of a planar graph in the plane so that its edges only meet at their ends. Let $G = (V, E, F)$ be a plane graph with the set of faces F. For a vertex $v \in V$, the *degree* and the *neighborhood* of v are denoted by $d(v)$ and $N(v)$, respectively. Call a vertex $v \in V$ a *k-vertex*, a *k^+-vertex*, or a *k^--vertex* if $d(v) = k$, $d(v) \geq k$, or $d(v) \leq k$, respectively. For a face $f \in F$, the set of vertices on f and the *boundary* walk of f are denoted by $V(f)$ and $b(f)$, respectively. The *size*, or more preferably here, the *degree* of f, denoted by $d(f)$, is the length of $b(f)$. The notions of a *k-face*, a *k^+-face*, or a *k^--face* are defined analogous to the ones of a k-vertex, a k^+-vertex, or a k^--vertex, respectively. Call a face *internal* if it is not the unbounded face. Call a vertex *external* if it is on the unbounded face; *internal* otherwise. For a face $f \in F$, the subgraph of G induced by $V(f)$ is denoted by $G[V(f)]$. If $u_1, u_2, \ldots, u_n$ are all vertices of $b(f)$ in a cyclic order, then we write $f = [u_1u_2 \ldots u_n]$. Two faces f and f' are *intersecting* if they have at least one vertex in common; *adjacent* if they have at least one edge xy in common, and we usually denote f' by f_{xy} when f and f' are adjacent with edge xy in common. Let C be a cycle of G. The *length* of C, denoted $|C|$, is the number of edges of C. A *k-cycle* is a cycle of length k. A *facial* 3-cycle, i.e., the boundary of a 3-face, is often called a *triangle*. A vertex or an edge is called *triangular* if it is incident with a triangle. If edge uv is non-triangular, then u is called an *isolated* neighbor of v. The set of vertices inside or outside a cycle C is denoted by $int(C)$ or $ext(C)$, respectively. Consequently, $Int(C) = G - ext(C)$ and $Ext(C) = G - int(C)$ are two vertex-induced subgraphs of G. Note that the chords of C lying inside C belong to $Ext(C)$. Call a cycle C *separating* if both $int(C)$ and $ext(C)$ are not empty. Sometimes, we do not distinguish C with $V(C)$ or $E(C)$.

Let $G = (V, E, F)$ be a plane graph without cycles of length 4 or 5, and C a cycle of length at most 9 in G. Call C *bad* if $Int(C)$ contains a subgraph H that is isomorphic to one of the configurations shown in Fig. 1, where C is the boundary of the unbounded face of the subgraph H. The subgraph H is called a *bad partition* of $Int(C)$, or simply C. Call a 9^--cycle *good* if it is not bad. By the definition of a bad cycle, if a cycle C is bad then $|C| = 8$ or 9. Note that all 7^--cycles are good.

A *chord* of a cycle C or a path P is an edge that connects two non-consecutive vertices of C or P. Let $e = xy$ be a chord of a cycle C, and P_1, P_2 the two paths of C between x and y. If the length of the cycle $C_i = P_i \cup \{e\}$ is k_i, $i = 1, 2$, then e is called a (k_1, k_2)*-chord* of C. Since G has no cycles of length 4 or 5, the following remark is obvious.

Remark 1. Let C be a cycle in G.
(1) If $|C| = 3, 6$, then C has no chord.
(2) If $|C| = 7$, then C has at most one chord, if any, a (3, 6)-chord.
(3) If $|C| = 8$, then C has at most two chords, if any, a (3, 7)-chord.
(4) If $|C| = 9$, then C has at most three chords, if any, a (3, 8)-chord.

2. Proof of Theorem 1

As early as in 1959, the coloring extension argument was first successfully applied by Grötzsch [10] to prove the well-known

Three Color Theorem. *Every planar graph without 3-cycles is 3-colorable, i.e.,* (0, 0, 0)*-colorable.*

Combining classical coloring extension argument with improper (defective) coloring, Xu [24] introduced the notion of *super extension* in improper colorings. In this paper, we will also employ the notion of super extension. Let G be a graph, H an induced subgraph of G, and ϕ a (2, 0, 0)-coloring of H. We say that ϕ can be *super extended* to G if ϕ can be extended to be a (2, 0, 0)-coloring of G, still denoted ϕ, so that $\phi(v) \neq \phi(u)$ whenever $u \in V(H)$, $v \in V(G) \setminus V(H)$ and $uv \in E(G)$.

Instead of proving Theorem 1, we will by the next two sections prove a stronger result as follows:

Theorem 2. *Let $G = (V, E, F)$ be a plane graph without cycles of length* 4 *or* 5*. If D, the boundary of the unbounded face of G, is a good cycle, then every* (2, 0, 0)*-coloring of $G[V(D)]$ can be super extended to the whole graph G.*

Assuming Theorem 2, we can easily derive Theorem 1:

Suppose to the contrary that Theorem 1 is false. Let G be a counterexample to Theorem 1 with the fewest vertices. By the Three Color Theorem above, G has at least one 3-cycle C. Embed G into the plane. If C is a separating cycle of G, giving a (2, 0, 0)-coloring of C, by the minimality of G, this (2, 0, 0)-coloring of C can be super extended to $Ext(C)$ and $Int(C)$