Discrete Mathematics 339 (2016) 906–922

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Updown categories: Generating functions and universal covers

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ARTICLE INFO

Article history:
Received 29 May 2013
Received in revised form 19 October 2015
Accepted 21 October 2015
Available online 14 November 2015

Keywords:
Category
Poset
Differential poset
Universal cover
Partition
Rooted tree

ABSTRACT

A poset can be regarded as a category in which there is at most one morphism between objects, and such that at most one of the sets $\mathrm{Hom}(c, c')$ and $\mathrm{Hom}(c', c)$ is nonempty for distinct objects c, c'. Retaining the latter axiom but allowing for more than one morphism between objects gives a sort of generalized poset in which there are multiplicities attached to the covering relations, and possibly nontrivial automorphism groups of objects. An updown category is such a category with an appropriate grading on objects. In this paper we give a precise definition of updown categories and develop a theory for them, including two types of associated generating functions and a notion of universal covers. We give a detailed account of ten examples, including updown categories of sets, graphs, necklaces, integer partitions, integer compositions, planar rooted trees, and rooted trees.

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1. Introduction

Suppose we have a collection of combinatorial objects, naturally graded, so that any object of rank n can be built up in n steps from a single object in rank 0. Further, for any object p of rank n and q of rank $n+1$, there are some number $u(p; q)$ of ways to build up p to make q, and $d(p; q)$ ways to break down q to get p. Here $u(p; q)$ and $d(p; q)$ are nonnegative integers, possibly unequal, though we require that $u(p; q) \neq 0$ if and only if $d(p; q) \neq 0$. For example (as in [8]) our collection might be the set of rooted trees, with $u(p; q)$ the number of vertices of the rooted tree p to which a new edge and terminal vertex can be added to get q, and $d(p; q)$ the number of terminal vertices (and incoming edges) that can be removed from q to get p.

We can obtain a natural definition of such a situation by modifying the categorical definition of a poset. A poset is usually thought of as a category with at most one morphism between objects, and at most one of the sets $\mathrm{Hom}(p, q)$ and $\mathrm{Hom}(p, q)$ nonempty when $p \neq q$. If we keep in place the second condition but permit $\mathrm{Hom}(p, q)$ to have more than one element, we allow for multiplicities (if $p \neq q$) and automorphisms (if $p = q$). If in addition the object set is graded, we call such a category (precisely defined in Section 2) an “updown category”. For an updown category $\mathcal{C}$, there are nonnegative integers $u(p; q)$ and $d(p; q)$ for $p, q \in \mathrm{Ob}\,\mathcal{C}$ with q having rank one greater than p, such that

$$u(p; q)|\mathrm{Aut}\, q| = d(p; q)|\mathrm{Aut}\, p|.$$

Then the set $\mathrm{Ob}\,\mathcal{C}$ has a natural graded poset structure, and the operators U and D on the free vector space $\Bbbk(\mathrm{Ob}\,\mathcal{C})$ defined by equations

$$Up = \sum_{\text{q covers p}} u(p; q)q \quad \text{and} \quad Dp = \sum_{\text{p covers q}} d(q; p)q$$

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http://dx.doi.org/10.1016/j.disc.2015.10.035
0012-365X/Published by Elsevier B.V.

are adjoint for the inner product on $\Bbbk(\operatorname{Ob}\mathcal{C})$ given by $\langle p, q\rangle = |\operatorname{Aut} p|\delta_{p,q}$. It is the existence of the operators U and D that inspires the name "updown". (The name comes from the same source as "down–up" in [2], namely Stanley's differential posets [14], but in [2] things are generalized in a different direction than in the present paper.)

For any updown category $\mathcal{C}$, there are associated two generating functions, defined in Section 3: the object generating function and the morphism generating function. If $\mathcal{C}$ is a univalent updown category (i.e., $u(p; q) = d(p; q)$ for all $p, q \in \operatorname{Ob}\mathcal{C}$), then the former is the rank-generating function of the graded poset $\operatorname{Ob}\mathcal{C}$. More generally, the object generating function of $\mathcal{C}$ can be thought of as a graded version of the cardinality of the underlying groupoid of $\mathcal{C}$ (see [1]). Computation of these generating functions is facilitated if $\mathcal{C}$ is evenly up-covered (i.e., $\sum_{q \text{ covers } p} u(p, q)$ depends only on the grade $|p|$ of p) or evenly down-covered ($\sum_{p \text{ covers } q} d(q; p)$ only depends on $|p|$).

Univalent updown categories admit a natural definition of universal covers. In [7] the author developed a theory of universal covers for weighted-relation posets, i.e., ranked posets in which each covering relation has a single number $n(x, y)$ assigned to it. The universal cover of a weighted-relation poset P is the "unfolding" of P into a usually much larger weighted-relation poset $\widetilde{P}$, so that the Hasse diagram of $\widetilde{P}$ is a tree and all covering relations of $\widetilde{P}$ have multiplicity 1. (This "unfolding" is a "simple" updown category in the sense defined in Section 1 below.) Although $\widetilde{P}$ had a natural description in each of the seven examples considered in [7], the general construction of $\widetilde{P}$ given in [7, Theorem 3.3] was somewhat unsatisfactory since it involved many arbitrary choices. In Section 4 we show that univalent updown categories are essentially categorified weighted-relation posets and give a functorial definition of universal covers for them (Theorem 4.3). We also give a functorial description of two univalent updown categories $\mathcal{C}^{\uparrow}$ and $\mathcal{C}^{\downarrow}$ associated with an updown category $\mathcal{C}$.

In Section 5 we offer ten examples, which encompass all those given in [7]. These include updown categories whose objects are the subsets of a finite set, monomials, necklaces, integer partitions, integer compositions, planar rooted trees, and rooted trees. For each example we compute the object and morphism generating functions and describe the associated covering spaces.

2. Updown categories

We begin by defining an updown category.

Definition 2.1. An updown category is a small category $\mathcal{C}$ with a rank functor $|\cdot| : \mathcal{C} \to \mathbb{N}$ (where $\mathbb{N}$ is the ordered set of natural numbers regarded as a category) such that

A1. Each rank $\mathcal{C}_n = \{p \in \operatorname{Ob}\mathcal{C} : |p| = n\}$ is finite.
A2. The zeroth rank $\mathcal{C}_0$ consists of a single object $\hat{0}$, and $\operatorname{Hom}(\hat{0}, p)$ is nonempty for all objects p of $\mathcal{C}$.
A3. For objects p, p' of $\mathcal{C}$, $\operatorname{Hom}(p, p')$ is always finite, and if $|p| = |p'|$ then $\operatorname{Hom}(p, p')$ is empty unless $p = p'$ (in which case $\operatorname{Hom}(p, p)$ is a group, denoted $\operatorname{Aut}(p)$).
A4. Any morphism $p \to p'$, where $|p'| = |p| + k$, factors as a composition $p = p_0 \to p_1 \to \cdots \to p_k = p'$, where $|p_{i+1}| = |p_i| + 1$;
A5. If $|p'| = |p| + 1$, the actions of $\operatorname{Aut}(p)$ and $\operatorname{Aut}(p')$ on $\operatorname{Hom}(p, p')$ (by precomposition and postcomposition respectively) are free.

From axiom A3 it follows that an updown category is skeletal (i.e., isomorphic objects are identical). We can define the multiplicities mentioned in the introduction as follows.

Definition 2.2. For any two objects p, p' of an updown category $\mathcal{C}$ with $|p'| = |p| + 1$, define

$$u(p; p') = \left|\operatorname{Hom}(p, p')/\operatorname{Aut}(p')\right| = \frac{|\operatorname{Hom}(p, p')|}{|\operatorname{Aut}(p')|}$$

and

$$d(p; p') = \left|\operatorname{Hom}(p, p')/\operatorname{Aut}(p)\right| = \frac{|\operatorname{Hom}(p, p')|}{|\operatorname{Aut}(p)|}.$$

It follows immediately from these definitions that

$$u(p; p')|\operatorname{Aut}(p')| = d(p; p')|\operatorname{Aut}(p)|. \tag{1}$$

We note two extreme cases. First, suppose $\mathcal{C}_n$ is empty for all $n > 0$. Then $\mathcal{C}$ is essentially the finite group $\operatorname{Aut}\hat{0}$. Second, suppose that every set $\operatorname{Hom}(p, p')$ has at most one element. Then $\mathcal{C}$ is a graded poset with least element $\hat{0}$.

Two important special types of updown categories are defined as follows.

Definition 2.3. An updown category $\mathcal{C}$ is univalent if $\operatorname{Aut}(p)$ is trivial for all $p \in \operatorname{Ob}\mathcal{C}$. An updown category $\mathcal{C}$ is simple if $\operatorname{Hom}(c, c')$ has at most one element for all $c, c' \in \operatorname{Ob}\mathcal{C}$, and the factorization in A4 is unique, i.e., for $|c'| > |c|$ any $f \in \operatorname{Hom}(c, c')$ has a unique factorization into morphisms between adjacent ranks.