Discrete Mathematics 339 (2016) 974–991

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

A generalization of carries process and riffle shuffles

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ARTICLE INFO

Article history:
Received 13 August 2014
Received in revised form 10 October 2015
Accepted 13 October 2015
Available online 14 November 2015

Keywords:
Carries process
Riffle shuffle

ABSTRACT

As a continuation to our previous work (Nakano and Sadahiro, 2014), we consider a generalization of carries process. Our results are: (i) right eigenvectors of the transition probability matrix, (ii) correlation of carries between different steps, and (iii) generalized riffle shuffle whose corresponding descent process has the same distribution as that of the generalized carries process.

1. Introduction

1.1. Background and definition

Carries process is the Markov chain of carries in adding array of numbers. It was Holte [6] who first studied the carries process, and he found many beautiful properties, e.g., the eigenvalues of the transition probability matrix P consist of negative powers of the base b, the eigenvectors of P are independent of b, and Eulerian numbers appear in the stationary distribution. Diaconis–Fulman [1–3] found connections to different subjects, e.g., the carries process has the same distribution as the descent process induced by the repeated applications of the riffle shuffle, and the array of the left eigenvectors of P coincides with the Foulkes character table of S_n.

In [8], we considered a generalization of the carries process in the sense that (i) we take various digit sets, and (ii) we also consider negative base. There we obtained (i) the left eigenvectors of P, and (ii) the limit theorem which yields the distribution of the sum of i.i.d. uniformly distributed random variables on $[0, 1]$. This paper is the continuation of [8]; here we study (i) the right eigenvectors of P, which yields the correlation of carries of different steps, and (ii) the equivalence to the descent process induced by the repeated generalized riffle shuffles on the colored permutation group. [5] is a review article of our results obtained so far.

In what follows, we first recall the definitions of carries process and some results in [8] (Section 1.2), and then state our results in this paper (Section 1.3). To simplify the statements, we shall discuss the positive/negative base simultaneously. Let $\pm b \in \mathbf{Z}(b \geq 2)$ be the base and let $\mathcal{D}_d := \{d, d+1, \ldots, d+b-1\}$ be the digit set such that $1-b \leq d \leq 0$. Then any $x \in \mathbf{N}$ has the unique representation

$$x = a_N(+b)^N + a_{N-1}(+b)^{N-1} + \cdots + a_0, \quad a_k \in \mathcal{D}_d,$$
$$x = a'_N(-b)^N + a'_{N-1}(-b)^{N-1} + \cdots + a'_0, \quad a'_k \in \mathcal{D}_d.$$

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http://dx.doi.org/10.1016/j.disc.2015.10.025
0012-365X/

In adding n numbers under this representation, let $C^{\pm}_{k-1}$ be the carry from the $(k-1)$th digit which belongs to a set $\mathcal{C}(\pm b, n)$ to be specified in Proposition 1.1. In the kth digit, we take $X_1, \dots, X_n$ uniformly at random from $\mathcal{D}_d$ and then the carry $C^{\pm}_k \in \mathcal{C}(\pm b, n)$ to the $(k+1)$th digit is determined by the following equation:

$$C^{\pm}_{k-1} + X_1 + \cdots + X_n = C^{\pm}_k \cdot (\pm b) + r, \quad r \in \mathcal{D}_d. \tag{1.1}$$

The process $\{C^{\pm}_k\}_{k=0}^{\infty}$ is Markovian with state space $\mathcal{C}(\pm b, n)$, which we call the n-**carries process over** $(\pm b, \mathcal{D}_d)$. Holte's carries process corresponds to the case where the base is positive and $d = 0$.

1.2. Our previous results

In this subsection we recall some results in [8] related to this paper. Let

$$l_{\pm} = l(\pm b, d) := \begin{cases} \dfrac{d}{b-1} & ((+b)\text{-case}) \\ -\dfrac{b+d}{b+1} & ((-b)\text{-case}). \end{cases}$$

Then the carry set $\mathcal{C}(\pm b, n)$ is explicitly given by Proposition 1.1.

Proposition 1.1.
(1) *The carry set $\mathcal{C}(\pm b, n)$ in the n-carries process over $(\pm b, \mathcal{D}_d)$ is given by*

$$\mathcal{C}(\pm b, n) = \{\min \mathcal{C}_{\pm}, \min \mathcal{C}_{\pm} + 1, \dots, \max \mathcal{C}_{\pm}\}$$
$$\min \mathcal{C}_{\pm} := \lfloor (n-1) l_{\pm} \rfloor, \quad \max \mathcal{C}_{\pm} := \lceil (n-1)(l_{\pm} + 1) \rceil.$$

(2) *The number of elements of $\mathcal{C}(\pm b, n)$ is*

$$\sharp \mathcal{C}(\pm b, n) = \begin{cases} n & ((n-1) l_{\pm} \in \mathbf{Z}) \\ n+1 & ((n-1) l_{\pm} \notin \mathbf{Z}). \end{cases}$$

$\sharp S$ is the number of elements of a finite set S. If $(n-1)l_{\pm} \notin \mathbf{Z}$, $\sharp \mathcal{C}(\pm b, n)$ is larger than that of Holte's carries process.

We introduce the following notation which is an important parameter to describe our results.

$$p = p(\pm b, d, n) := \frac{1}{1 - \langle (n-1) l_{\pm} \rangle} = \begin{cases} 1 & ((n-1) l_{\pm} \in \mathbf{Z}) \\ \langle (n-1)(-l_{\pm}) \rangle^{-1} & ((n-1) l_{\pm} \notin \mathbf{Z}) \end{cases}$$

where $\langle x \rangle := x - \lfloor x \rfloor$ is the fractional part of x.

Remark 1.1. (1) $p \geq 1$ is rational and $p = 1$ if and only if $\sharp \mathcal{C}(b, n) = n$, including Holte's carries process as a special case.

(2) Proposition 1.1 can be understood from the following consideration. Let $F := \left\{ \frac{a_1}{b} + \cdots + \frac{a_N}{b^N} \mid N \in \mathbf{N},\ a_1, \dots, a_N \in \mathcal{D}_d \right\}$. which is dense in $(l_{\pm}, l_{\pm} + 1)$. Then $c \in \mathcal{C}(\pm b, n)$ if and only if $\overbrace{(F + \cdots + F)}^{n} \cap (c + F) \neq \emptyset$. If $(n-1) l_{\pm} \notin \mathbf{Z}$, then we need $(n+1)$- c's to cover $\overbrace{F + \cdots + F}^{n}$ by the sets of the form $c + F$ so that $\sharp \mathcal{C}(\pm b, n) = n + 1$. p is defined such that $(n-1) l_{\pm} - \lfloor (n-1) l_{\pm} \rfloor = 1 - \frac{1}{p}$.

To study further, let us change variables in (1.1): $X_j = Y_j + d$, $r = s + d$, $C^{\pm}_k = \kappa^{\pm}_k + \min \mathcal{C}_{\pm}$ so that

$$Y_j, s \in \mathcal{D}(b) := \{0, 1, \dots, b-1\},$$
$$\kappa^{\pm}_k \in \mathcal{C}_p(n) := \begin{cases} \{0, 1, \dots, n-1\} & (p = 1) \\ \{0, 1, \dots, n\} & (p \neq 1). \end{cases}$$

Substituting them into (1.1), we have

$$\begin{cases} \kappa^{+}_{k-1} + Y_1 + \cdots + Y_n + A_{+}(b) = \kappa^{+}_k b + s & ((+b)\text{-case}) \\ \kappa^{-}_{k-1} + Y_1 + \cdots + Y_n + A_{-}(b) = (n - \kappa^{-}_k) b + s & ((-b)\text{-case}) \end{cases}$$
$$\text{where } A_{+}(b) := \frac{b-1}{p^*}, A_{-}(b) := \frac{b+1}{p} - 1, \frac{1}{p} + \frac{1}{p^*} = 1. \tag{1.2}$$

It says that, $\{\kappa^{\pm}_k\}$ is similar to $(n+1)$-carries process over $(b, \mathcal{D}_0)$ except that $X_{n+1} = A_{\pm}(b)$ and C^{-}_k in (1.1) is replaced by $n - C^{-}_k$. Note that $A_{\pm}(b) \in \mathbf{N}$. Conversely, if $p \geq 1$ and $\frac{b \mp 1}{p} \in \mathbf{N}$ (equivalently, $b \equiv \pm 1 \pmod{p}$ if $p \in \mathbf{N}$), then (1.2) still defines a Markov chain $\{\kappa^{\pm}_k\}$ on $\mathcal{C}_p(n)$, even if it does not correspond to a n-carries process over $(\pm b, \mathcal{D}_d)$ for some d. We