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Planar graphs without 5-cycles and intersecting triangles are (1, 1, 0)-colorable

Runrun Liu [a], Xiangwen Li [a], Gexin Yu [a,b,*]

[a] Department of Mathematics, Huazhong Normal University, Wuhan, 430079, China

[b] Department of Mathematics, The College of William and Mary, Williamsburg, VA, 23185, USA

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ABSTRACT

A $(c_1, c_2, \ldots, c_k)$-coloring of G is a mapping $\varphi : V(G) \mapsto \{1, 2, \ldots, k\}$ such that for every i, $1 \le i \le k$, $G[V_i]$ has maximum degree at most c_i, where $G[V_i]$ denotes the subgraph induced by the vertices colored i. Borodin and Raspaud conjecture that every planar graph without 5-cycles and intersecting triangles is (0, 0, 0)-colorable. We prove in this paper that such graphs are (1, 1, 0)-colorable.

1. Introduction

Graph coloring is one of the central topics in graph theory. A graph is $(c_1, c_2, \ldots, c_k)$-*colorable* if the vertex set can be partitioned into k sets $V_1, V_2, \ldots, V_k$, such that for every $i : 1 \le i \le k$ the subgraph $G[V_i]$ has maximum degree at most c_i. Thus a (0, 0, 0)-colorable graph is properly 3-colorable.

The problem of deciding whether a planar graph is properly 3-colorable is NP-complete. Significant research has been devoted to finding conditions for a planar graph to be properly 3-colorable. The well-known Grötzsch Theorem [8] shows that "triangle-free" suffices. The famous Steinberg Conjecture [13] proposes that "free of 4-cycles and 5-cycles" is also enough.

Conjecture 1.1 (*Steinberg, [13]*). *All planar graphs without 4-cycles and 5-cycles are 3-colorable.*

Some relaxations of the Steinberg Conjecture are known to be true. Along the direction suggested by Erdős to find a constant c such that a planar graph without cycles of length from 4 to c is 3-colorable, Borodin, Glebov, Raspaud, and Salavatipour [4] showed that $c \le 7$, and more results similar to those can be found in the survey by Borodin [1]. Another direction of relaxation of the conjecture is to allow some defects in the color classes. Chang, Havet, Montassier, and Raspaud [6] proved that all planar graphs without 4-cycles or 5-cycles are (2, 1, 0)-colorable and (4, 0, 0)-colorable. In [10,11,16], it is shown that planar graphs without 4-cycles or 5-cycles are (3, 0, 0)- and (1, 1, 0)-colorable. Some more results along this direction can be found in the papers by Wang et al. [16,17].

Havel [9] proposed that planar graphs with triangles far apart should be properly 3-colorable, which was confirmed in a preprint of Dvořák, Král and Thomas [7]. Borodin and Raspaud [5] combined the ideas of Havel and Steinberg and proposed the following so called Bordeaux Conjecture in 2003.

* Corresponding author at: Department of Mathematics, The College of William and Mary, Williamsburg, VA, 23185, USA.
E-mail addresses: xwli68@mail.ccnu.edu.cn (X. Li), gyu@wm.edu (G. Yu).

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Conjecture 1.2 (*Borodin and Raspaud, [5]*). *Every planar graph without intersecting triangles and without 5-cycles is 3-colorable.*

A planar graph without intersecting triangles means the distance between triangles is at least 1. Let $d^{\triangledown}$ denote the smallest distance between any pair of triangles in a planar graph. A relaxation of the Bordeaux Conjecture with $d^{\triangledown} \geq 4$ was confirmed by Borodin and Raspaud [5], and the result was improved to $d^{\triangledown} \geq 3$ by Borodin and Glebov [2] and, independently, by Xu [14]. Borodin and Glebov [3] further improved the result to $d^{\triangledown} \geq 2$.

Using the relaxed coloring notation, Xu [15] proved that all planar graphs without adjacent triangles and 5-cycles are (1, 1, 1)-colorable, where two triangles are adjacent if they share an edge.

Let $\mathcal{G}$ be the family of plane graphs with $d^{\triangledown} \geq 1$ and without 5-cycles. Yang and Yerger [18] showed that planar graphs in $\mathcal{G}$ are (4, 0, 0)- and (2, 1, 0)-colorable, but there is a flaw in one of their key lemmas (Lemma 2.4). In [12], we showed that graphs in $\mathcal{G}$ are (2, 0, 0)-colorable.

In this paper, we will prove another relaxation of the Bordeaux Conjecture. Let G be a graph and H be a subgraph of G. We call (G, H) to be *superextendable* if each (1, 1, 0)-coloring of H can be extended to G so that vertices in $G - H$ have different colors from their neighbors in H; in this case, we call H to be a superextendable subgraph.

Theorem 1.3. *Every triangle or 7-cycle of a planar graph in $\mathcal{G}$ is superextendable.*

As a corollary, we have the following relaxation of the Bordeaux Conjecture.

Theorem 1.4. *A planar graph in $\mathcal{G}$ is (1, 1, 0)-colorable.*

To see the truth of Theorem 1.4 by way of Theorem 1.3, we may assume that the planar graph contains a triangle C since G is (0, 0, 0)-colorable if G has no triangle. Then color the triangle, and by Theorem 1.3, the coloring of C can be superextended to G. Thus, we get a coloring of G.

As many results with similar fashion, we use a discharging argument to prove Theorem 1.3. This argument consists of two parts: structures and discharging. After introducing some common notation in Section 2, we show in Section 3 some useful special structures in a minimal counterexample to the theorem, then in Section 4, we design a discharging process to distribute the charges and use the special structures to reach a contradiction.

It should be noted that while the proof of our main theorem shares a lot of common properties with the (2, 0, 0) result in [12], it is much more involved. We have to extend some powerful tools from [15] by Xu, and discuss in detail the structures around 4-vertices and 5-vertices. It would be interesting to know how to use the new tools developed in this paper to get better results towards the Bordeaux Conjecture.

2. Preliminaries

In this section, we introduce some notation used in the paper.

Graphs mentioned in this paper are all simple. For a positive integer n, let $[n] = \{1, 2, \ldots, n\}$. A k-vertex (k^+-vertex, k^--vertex) is a vertex of degree k (at least k, at most k). The same notation will apply to faces and cycles. We use $b(f)$ to denote the vertex sets on f. We use $F(G)$ to denote the set of faces in G. An $(l_1, l_2, \ldots, l_k)$-face is a k-face $v_1v_2 \ldots v_k$ with $d(v_i) = l_i$, respectively. A face f is a *pendant 3-face* of vertex v if v is not on f but is adjacent to some 3-vertex on f. A *pendant neighbor* of a 3-vertex v on a 3-face is the neighbor of v not on the 3-face.

Let C be a cycle of a plane graph G. We use $int(C)$ and $ext(C)$ to denote the sets of vertices located inside and outside C, respectively. The cycle C is called a *separating cycle* if $int(C) \neq \emptyset \neq ext(C)$, and is called a *nonseparating cycle* otherwise. We still use C to denote the set of vertices of C.

Let $S_1, S_2, \ldots, S_l$ be pairwise disjoint subsets of $V(G)$. We use $G[S_1, S_2, \ldots, , S_l]$ to denote the graph obtained from G by identifying all the vertices in S_i to a single vertex for each $i \in \{1, 2, \ldots, l\}$.

A vertex v is *properly colored* if all neighbors of v have different colors from v. A vertex v is *nicely colored* if it shares a color (say i) with at most $\max\{s_i - 1, 0\}$ neighbors, where s_i is the deficiency allowed for color i; thus if a vertex v is nicely colored by a color i which allows deficiency $s_i > 0$, then an uncolored neighbor of v can be colored by i.

3. Special configurations

Let (G, C_0) be a minimum counterexample to Theorem 1.3 with minimum $\sigma(G) = |V(G)| + |E(G)|$, where C_0 is a triangle or a 7-cycle in G that is precolored. For simplicity, let $F_k = \{f : f \text{ is a } k\text{-face and } b(f) \cap C_0 = \emptyset\}$, $F'_k = \{f : f \text{ is a } k\text{-face and } |b(f) \cap C_0| = 1\}$, and $F''_k = \{f : f \text{ is a } k\text{-face and } |b(f) \cap C_0| = 2\}$.

The following lemmas are shown in [12].

Proposition 3.1 (*Prop 3.1 in [12]*). *(a) Every vertex not on C_0 has degree at least 3.*

(b) A k-vertex in G can have at most one incident 3-face.

(c) No 3-face and 4-face in G can have a common edge.

Lemma 3.2 (*Lemma 3.2 in [12]*). *The graph G contains neither separating triangles nor separating 7-cycles.*