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Forbidden substructure for interval digraphs/bigraphs

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ABSTRACT

An interval matrix is the adjacency matrix of an interval digraph or equivalently the biadjacency matrix of an interval bigraph. In this paper we investigate the forbidden substructures of an interval bigraph. Our method finds hitherto existing forbidden substructures for interval matrices, and via a more concise statement, as well as a new example showing that these substructures are not exhaustive.

1. Introduction

An *interval digraph* is a directed graph representable by assigning each vertex v an ordered pair (S_v, T_v) of closed intervals on a line so that uv is a (directed) edge if and only if S_u intersects T_v. An *interval bigraph* is a bipartite graph representable by assigning each vertex v an interval so that vertices in opposite partite sets are adjacent if and only if their intervals intersect. The *biadjacency matrix* (also called *reduced adjacency matrix*) of a bipartite graph is the submatrix of its adjacency matrix consisting of the rows indexed by one partite set and the columns indexed by the other.

Interval bigraphs and interval digraphs were introduced in [13] and [23] respectively. As observed in [18] and [25], the two concepts are equivalent. The point is that the adjacency matrix of an interval digraph is the biadjacency matrix of an interval bigraph and conversely the biadjacency matrix of an interval bigraph becomes the adjacency matrix of an interval digraph, by adding, if necessary, rows or columns of 0s to make it square.

Several characterizations of interval bigraph are known (see [14,22,23]). One characterization uses *Ferrers bigraphs* (introduced by Guttman [12] and by Riguet [20]), which are bigraphs satisfying any of the following equivalent conditions.

(A) The set of neighbors are linearly ordered by inclusion.

(B) The rows and the columns of the biadjacency matrix can be permuted (independently) so that the 1s cluster in the upper right (or, lower left) as a Ferrers diagram.

(C) The biadjacency matrix has no 2-by-2 permutation matrix as a submatrix.

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The condition in terms of Ferrers bigraphs similarly extends to arbitrary binary matrices. A binary matrix with no 2-by-2 permutation submatrix is a *Ferrers matrix*.

An example of a Ferrers matrix is

$$\begin{array}{|cccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ \hline \end{array}$$

Theorem A characterizes interval bigraphs. We forbid multiple edges, so a bigraph is *complete* if every entry in its biadjacency matrix is 1. A *zero-partition* of a binary matrix is a coloring of each 0 with R or C in such a way that every R has only 0s colored R to its right and every C has only 0s colored C below it. A matrix that admits a zero partition after suitable row and column permutations is *zero-partitionable*.

Theorem A (*[23]*)**.** *The following conditions are equivalent*

(i) *B is an interval bigraph.*
(ii) *The rows and columns of its biadjacency matrix can be independently permuted such that the matrix admits a zero partition.*
(iii) *B is the intersection of two Ferrers bigraphs whose union is complete.*

We defined zero-partitionability for general binary matrices. Adding rows or columns of 0s does not affect zero-partitionability, so this property of the biadjacency matrix also characterizes interval bigraphs (our bigraphs are simple). We discuss interval bigraphs wholly in terms of the binary biadjacency matrices, so we call such a matrix an *interval matrix*. Our aim is to provide forbidden substructures of interval matrices.

In the present paper, we obtain five small matrix configurations, each of which give rise to a set of forbidden matrices for interval matrices. Interestingly, along the way, all the forbidden subgraphs for interval bigraphs, so far known, are subsumed in our approach; in addition, new forbidden substructures are obtained.

More results on interval digraphs/bigraphs and related topics appear in [2,16,25,27]. For a remarkably fine survey of the topic of the interval bigraphs and related areas, see Brown [1].

1.1. Background materials

A binary matrix with exactly one 0 is a Ferrers matrix. Hence every bigraph B is the intersection of finitely many Ferrers bigraphs. The minimum number of Ferrers bigraphs whose intersection is B is the *Ferrers dimension* of the bigraph B, written $fdim(B)$. More generally, the Ferrers dimension of a binary matrix A is the minimum number of Ferrers matrices whose intersection is A.

Ferrers bigraphs and Ferrers dimension were extensively studied, in the language of digraphs, by Cogis in [4–6]. For short proofs of equivalence of various characterizations of Ferrers bigraphs, see [27]. Mahadev and Peled in their book on Threshold Graphs [17] devoted half a chapter on Ferrers digraphs. Golumbic and Trenk [11] found it important to write a section on Ferrers dimension two in their book on Tolerance Graphs. Ferrers bigraphs and Ferrers dimension are also called (bipartite) chain graphs and chain dimension respectively. The term chain graph was first used by Yannakakis [28] in his paper on partial order dimension problem.

By Theorem A, every interval bigraph has Ferrers dimension at most 2. The converse is false [23]. Bigraphs with Ferrers dimension 2 were characterized by Cogis [4] and by Doignon, Ducamp, and Falmagne [9]. Cogis [4] introduced the *associated graph* $H(B)$ for a bigraph B. Its vertices are the 0s of the biadjacency matrix $A(B)$ of B, with two such vertices adjacent in $H(B)$ if and only if they are the 0s of a 2-by-2 permutation submatrix of $A(B)$. Such a submatrix is an *obstruction* and two zeros form obstructions with one another. Cogis [4] proved the following theorem.

Theorem B. *The Ferrers dimension of a bipartite graph B is at most 2 if and only if* $H(B)$ *is bipartite.*

This yields a fast algorithm for testing $fdim(B) \leq 2$.

The associated graph $H(A)$ is defined in the same way for a general binary matrix A. Entries in a row or column of 0s become isolated vertices in $H(A)$, which does not affect whether $H(A)$ is bipartite. Thus the characterization of $fdim(B) \leq 2$ extends to binary matrices. For more equivalent notions of Ferrers dimension two, and other related results, see [3,21,23,24].

Partial results on forbidden configuration for interval matrices were obtained in [7]. In Section 2, we introduce the necessary concepts and state our forbidden substructures, which strengthens the results of [7].

Müller [18] also considered this from a different approach. For this he defined asteroidal triple of edges (ATE) in a bigraph as follows : Three edges e_1, e_2, e_3 form an *asteroidal triple of edges* if there is a path joining two edges that avoids the neighbors of the third. Müller also proved that an interval bigraph is ATE-free. Das and Sen [8] showed that a bigraph B having Ferrers dimension less than equal to 2 is also ATE-free, which strengthens the Müller's result.