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Characterizing forbidden pairs for rainbow connection in graphs with minimum degree 2

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ABSTRACT

A connected edge-colored graph G is rainbow-connected if any two distinct vertices of G are connected by a path whose edges have pairwise distinct colors; the rainbow connection number $\mathrm{rc}(G)$ of G is the minimum number of colors that are needed in order to make G rainbow connected. In this paper, we complete the discussion of pairs (X, Y) of connected graphs for which there is a constant k_{XY} such that, for every connected (X, Y)-free graph G with minimum degree at least 2, $\mathrm{rc}(G) \leq \mathrm{diam}(G) + k_{XY}$ (where $\mathrm{diam}(G)$ is the diameter of G), by giving a complete characterization. In particular, we show that for every connected $(Z_3, S_{3,3,3})$-free graph G with $\delta(G) \geq 2$, $\mathrm{rc}(G) \leq \mathrm{diam}(G) + 156$, and, for every connected $(S_{2,2,2}, N_{2,2,2})$-free graph G with $\delta(G) \geq 2$, $\mathrm{rc}(G) \leq \mathrm{diam}(G) + 72$.
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1. Introduction

We use [2] for terminology and notation not defined here and consider finite simple undirected graphs only. To avoid trivial cases, all graphs considered will be connected with at least one edge.

A subgraph of an edge-colored graph G is *rainbow* if all its edges have pairwise distinct colors, and G is *rainbow-connected* if, for any $x, y \in V(G)$, the graph G contains a rainbow path with x, y as endvertices. Note that the edge coloring need not be proper. The *rainbow connection number* of G, denoted by $\mathrm{rc}(G)$, is the minimum number of colors that are needed in order to make G rainbow connected.

This concept of rainbow connection in graphs was introduced by Chartrand et al. in [5]. An easy observation is that if G has n vertices then $\mathrm{rc}(G) \leq n - 1$, since one may color the edges of a given spanning tree of G with different colors and color the remaining edges with one of the already used colors. Chartrand et al. determined the precise value of the rainbow connection number for several graph classes including complete multipartite graphs [5]. The rainbow connection number has been studied for further graph classes in [3,7,11,15] and for graphs with fixed minimum degree in [3,12,17]. See [16] for a survey.

There are various applications for such edge colorings of graphs. One interesting example is the secure transfer of classified information between agencies (see, e.g., [8]).

For the rainbow connection number of graphs, the following results are known (and obvious).

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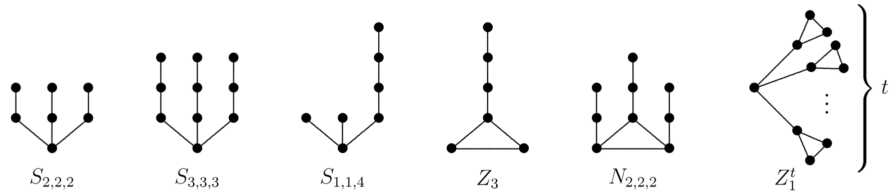

Fig. 1. The graphs $S_{2,2,2}, S_{3,3,3}, S_{1,1,4}, Z_3, N_{2,2,2}$ and Z_1^t.

Proposition A. *Let G be a connected graph of order n. Then*

(i) $1 \le \mathrm{rc}(G) \le n-1$,
(ii) $\mathrm{rc}(G) \ge \mathrm{diam}(G)$,
(iii) $\mathrm{rc}(G) = 1$ *if and only if G is complete,*
(iv) $\mathrm{rc}(G) = n-1$ *if and only if G is a tree.*

Note that the difference $\mathrm{rc}(G) - \mathrm{diam}(G)$ can be arbitrarily large since e.g. for $G \simeq K_{1,n-1}$ we have $\mathrm{rc}(K_{1,n-1}) - \mathrm{diam}(K_{1,n-1}) = (n-1) - 2 = n-3$. Especially, each bridge requires a single color. For bridgeless graphs, the following upper bound is known.

Theorem B ([1]). *For every connected bridgeless graph G with radius r,*

$$\mathrm{rc}(G) \le r(r+2).$$

Moreover, for every integer $r \ge 1$, there exists a bridgeless graph G with radius r and $\mathrm{rc}(G) = r(r+2)$.

Note that this upper bound is still quadratic in terms of the diameter of G.

Let $\mathcal{F}$ be a family of connected graphs. We say that a graph G is $\mathcal{F}$*-free* if G does not contain an induced subgraph isomorphic to a graph from $\mathcal{F}$. Specifically, for $\mathcal{F} = \{X\}$ we say that G is X*-free*, and for $\mathcal{F} = \{X, Y\}$ we say that G is (X, Y)*-free*. The members of $\mathcal{F}$ will be referred to in this context as *forbidden induced subgraphs*.

If X_1, X_2 are graphs, we write $X_1 \overset{\mathrm{IND}}{\subset} X_2$ if X_1 is an induced subgraph of X_2 (not excluding the possibility that $X_1 = X_2$), and if $\{X_1, Y_1\}, \{X_2, Y_2\}$ are pairs of graphs, we write $\{X_1, Y_1\} \overset{\mathrm{IND}}{\subset} \{X_2, Y_2\}$ if either $X_1 \overset{\mathrm{IND}}{\subset} Y_1$ and $X_2 \overset{\mathrm{IND}}{\subset} Y_2$, or $X_1 \overset{\mathrm{IND}}{\subset} Y_2$ and $X_2 \overset{\mathrm{IND}}{\subset} Y_1$. It is straightforward to see that if $X_1 \overset{\mathrm{IND}}{\subset} X_2$, then every X_1-free graph is X_2-free, and if $\{X_1, Y_1\} \overset{\mathrm{IND}}{\subset} \{X_2, Y_2\}$, then every (X_1, Y_1)-free graph is (X_2, Y_2)-free.

Although, by Theorem B, $\mathrm{rc}(G)$ can be quadratic in terms of $\mathrm{diam}(G)$, it turns out that forbidden subgraph conditions can remarkably lower the upper bound on $\mathrm{rc}(G)$. In [9], the authors considered the question for which families $\mathcal{F}$ of connected graphs, a connected $\mathcal{F}$-free graph satisfies $\mathrm{rc}(G) \le \mathrm{diam}(G) + k_{\mathcal{F}}$, where $k_{\mathcal{F}}$ is a constant (depending on $\mathcal{F}$), and gave a complete answer for $1 \le |\mathcal{F}| \le 2$ by the following two results (where N denotes the *net*, i.e. the graph obtained by attaching a pendant edge to each vertex of a triangle).

Theorem C ([9]). *Let X be a connected graph. Then there is a constant k_X such that every connected X-free graph G satisfies* $\mathrm{rc}(G) \le \mathrm{diam}(G) + k_X$*, if and only if $X = P_3$.*

Theorem D ([9]). *Let X, Y be connected graphs, $X, Y \ne P_3$. Then there is a constant k_{XY} such that every connected (X, Y)-free graph G satisfies* $\mathrm{rc}(G) \le \mathrm{diam}(G) + k_{XY}$*, if and only if either $\{X, Y\} \overset{IND}{\subset} \{K_{1,r}, P_4\}$ for some $r \ge 4$, or $\{X, Y\} \overset{IND}{\subset} \{K_{1,3}, N\}$.*

Let $S_{i,j,k}$ denote the graph obtained by identifying one endvertex of three vertex disjoint paths of lengths i, j, k, Z_i the graph obtained by attaching a path of length i to a vertex of a triangle, and let $N_{i,j,k}$ denote the graph obtained by identifying each vertex of a triangle with an endvertex of one of three vertex disjoint paths of lengths i, j, k. We also use Z_1^t to denote the graph obtained by attaching a triangle to each vertex of degree 1 of a star $K_{1,t}$ (see Fig. 1).

In [10], the authors considered an analogous question for graphs with minimum degree at least two and gave the following results.

Theorem E ([10]). *Let X be a connected graph. Then there is a constant k_X such that every connected X-free graph G with $\delta(G) \ge 2$ satisfies* $\mathrm{rc}(G) \le \mathrm{diam}(G) + k_X$*, if and only if $X \overset{IND}{\subset} P_5$.*

Theorem F ([10]). *Let $X, Y \overset{IND}{\not\subset} P_5$ be a pair of connected graphs for which there is a constant k_{XY} such that every connected (X, Y)-free graph G with $\delta(G) \ge 2$ satisfies* $\mathrm{rc}(G) \le \mathrm{diam}(G) + k_{XY}$*. Then $\{X, Y\} \overset{IND}{\subset} \{P_6, Z_1^r\}$ for some $r \in \mathbb{N}$, or $\{X, Y\} \overset{IND}{\subset} \{Z_3, P_7\}$, or $\{X, Y\} \overset{IND}{\subset} \{Z_3, S_{1,1,4}\}$, or $\{X, Y\} \overset{IND}{\subset} \{Z_3, S_{3,3,3}\}$, or $\{X, Y\} \overset{IND}{\subset} \{S_{2,2,2}, N_{2,2,2}\}$.*

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