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Strategic balance in graphs[☆,☆☆]

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ABSTRACT

For a given graph G, a nonempty subset S contained in $V(G)$ is an *alliance* iff for each vertex $v \in S$ there are at least as many vertices from the closed neighbourhood of v in S as in $V(G) - S$. An alliance is *global* if it is also a dominating set of G. The *alliance partition number* of G was defined in Hedetniemi et al. (2004) to be the maximum number of sets in a partition of $V(G)$ such that each set is an alliance. Similarly, in Eroh and Gera (2008) the *global alliance partition number* is defined for global alliances, where the authors studied the problem for (binary) trees.

In the paper we introduce a new concept of *strategic balance* in graphs: for a given graph G, determine whether there is a partition of vertex set $V(G)$ into three subsets N, S and I such that both N and S are global alliances. We give a survey of its general properties, e.g., showing that a graph G has a strategic balance iff its global alliance partition number equals at least 2. We construct a linear time algorithm for solving the problem in trees (thus giving an answer to the open question stated in Eroh and Gera (2008)) and studied this problem for many classes of graphs: paths, cycles, wheels, stars, complete graphs and complete k-partite graphs. Moreover, we prove that this problem is $\mathcal{NP}$-complete for graphs with a degree bounded by 4 and state an open question regarding subcubic graphs.

1. Introduction

1.1. Problem definition

In the following we consider solely simple connected finite non-empty graphs, and we use standard notation of graph theory. Let G be a graph where $V(G)$ is a set of vertices and $E(G)$ is a set of edges. By Δ we denote the maximum degree in graph. For each vertex $v \in V(G)$ sets $N(v) = \{u \in V(G) : \{u, v\} \in E\}$ and $N[v] = N(v) \cup \{v\}$ are *open* and *closed neighbourhood* of vertex v, respectively. Similarly, for a subset $X \subset V(G)$ sets $N(X) = \bigcup_{u \in X} N(u)$ and $N[X] = N(X) \cup X$ are *open* and *closed neighbourhood* of set X, respectively.

In this paper we study the problem of strategic balance. The problem is connected to the problem of global alliance, which was introduced in [6] and [7]. For a graph G, a nonempty subset S contained in $V(G)$ is an *alliance* if for each vertex $v \in S$

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there are at least as many vertices from the closed neighbourhood of v in S as in $V(G) - S$, i.e. $|N[v] \cap S| \geq |N[v] - S|$. An alliance is *global* if it is also a dominating set of G, i.e. $N[S] = V$.

Definition 1. For a partition of $V(G)$ into three sets N, S, I where N and S are global alliances the pair $(\{N, S\}, I)$ is a strategic balance. If I is an empty set, the strategic balance is perfect and denoted as $\{N, S\}$.

Intuitively sets N and S can be seen as two sides of the conflict, for example **N**orth and **S**outh. Both sets are safe in terms of the alliance and they have the same global scope i.e. each global alliance has an access to the whole graph (dominating property). Therefore, there is no essential difference between sets N and S. Set I is formed by intermediate vertices which have not chosen any side of the conflict. We cannot assume that they will support or attack any side of the conflict. Sets N and S must be prepared for the worst case thus both consider members of set I as potential enemies.

The question whether there exists a strategic balance in a graph is *Strategic Balance* ($\mathcal{SB}$) problem. Similarly, we define *Perfect Strategic Balance* ($\mathcal{PSB}$) problem as the question whether there exists a perfect strategic balance in a graph.

1.2. Related problems and our contribution

The global alliance problem was naturally motivated as a model of conflict situations. Examples of problems such as alliances between people, countries or plants in botany are mentioned in [2]. The strategic balance problem models the situation when the two sides of the conflict are present.

In [7] the authors defined the *alliance partition number* of graph G to be the maximum number of sets in a partition of $V(G)$ such that each set is an alliance. Similarly, the *global alliance partition number* is defined for global alliances in [3], where the authors studied the problem in trees. By definition, a strategic balance is close to the case when the global alliance partition number equals at least 2. In fact, we proved that these problems are equivalent.

In [3] the authors studied the global alliance partition number problem in trees and observed that this number is equal to 1 or 2 in a tree. They gave partial results regarding the characterization of the trees that cannot have two disjoint global alliances and left the problem of full characterization open. In this paper we give a linear time algorithm that finds the global alliance partition number of a tree, and constructs the partition. We also solved the problem for complete graphs and complete k-partite graphs, as well as basic classes: paths, cycles, wheels and stars.

In the paper [10] the authors proved that the problem of partition into 2 global alliances is $\mathcal{NP}$-complete. We improve this result and show that this problem is $\mathcal{NP}$-complete even for graphs with $\Delta \leq 4$. The case $\Delta \leq 3$ is still open, and we conjecture that it can be solved in polynomial time.

2. Strategic balance and perfect strategic balance

The main result in this section is the equivalence between the existence of a strategic balance and a perfect strategic balance in a graph.

In the following, for the sake of notation simplicity, we define $Av \stackrel{df}{=} N[v] \cap A$. Let G be a graph.

Proposition 1. *For each $A \subseteq B \subseteq V(G)$ and for each $v \in V(G)$ we have $|Bv| \geq |Av|$.*

Proof. Since $A \subseteq B$, we have $|Bv| = |N[v] \cap B| \geq |N[v] \cap A| = |Av|$. □

Proposition 2. *For every two (global) alliances A and B in graph G we have that $A \cup B$ is a (global) alliance in G.*

Proof. Let $v \in A \cup B$, and without loss of generality let $v \in A$. By Proposition 1 we have $|(A \cup B)v| \geq |Av| \geq |(V(G) - A)v| \geq |(V(G) - (A \cup B))v|$. If A and B are the dominating sets of G, then obviously, $A \cup B$ is the dominating set of G. □

Proposition 3. *There is a perfect strategic balance in a graph G iff its global alliance partition number equals at least* 2.

Proof. If there is a perfect strategic balance in a graph, then we have two disjoint global alliances which cover the whole vertex set. Let us assume that we have a partition of vertex set $V(G)$ into global alliances $S_1, \ldots, S_k$, where $k \geq 2$. By Proposition 2 we have that $\bigcup_{i=1}^{k-1} S_i$ is the global alliance, thus $\bigcup_{i=1}^{k-1} S_i$ and S_k is the partition of $V(G)$ into global alliances. □

For a given graph G and a global alliance $N \subseteq V(G)$ the question whether there exists a subset $S' \subseteq (V(G) - N)$ such that there exists a strategic balance $(\{N', S'\}, I)$, where $N \subseteq N'$ is referred to as *Strategic Balance Opponent* ($\mathcal{SBO}$) problem. If we additionally require $I = \emptyset$, we refer to this problem as *Perfect Strategic Balance Opponent* ($\mathcal{PSBO}$) problem.

Lemma 4. *Let $N \subseteq V(G)$ be a global alliance in a graph G. For each $v \in V(G) - N$, if $|Nv| > |(V(G) - N)v|$, then there is no alliance $S \subseteq V(G) - N$ containing vertex v. Moreover, $N \cup \{v\}$ is a global alliance.*

Proof. Let $v \in V(G) - N$ and $|Nv| > |(V(G) - N)v|$. Let us assume, to the contrary, that there is an alliance $S \subseteq V(G) - N$ such that $v \in S$. Hence, by an alliance property we have $|Sv| \geq |(V(G) - S)v|$ and by Proposition 1 we have $|(V(G) - N)v| \geq |Sv|$.