Discrete Mathematics 339 (2016) 1848–1860

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

# Interval cyclic edge-colorings of graphs

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ARTICLE INFO

*Article history:*
Received 2 November 2014
Accepted 30 January 2016
Available online 2 March 2016

*Keywords:*
Edge-coloring
Interval coloring
Interval cyclic coloring
Bipartite graph
Complete graph

ABSTRACT

A proper edge-coloring $\alpha$ of a graph $G$ with colors $1, \ldots, t$ is called an *interval cyclic $t$-coloring* if all colors are used, and the colors of edges incident to each vertex $v$ of $G$ either form an interval of integers or the set $\{1, \ldots, t\} \setminus \{\alpha(e)\colon e \text{ is incident to } v\}$ is an interval of integers. A graph $G$ is *interval cyclically colorable* if it has an interval cyclic $t$-coloring for some positive integer $t$. The set of all interval cyclically colorable graphs is denoted by $\mathfrak{N}_c$. For a graph $G \in \mathfrak{N}_c$, the least and the greatest values of $t$ for which it has an interval cyclic $t$-coloring are denoted by $w_c(G)$ and $W_c(G)$, respectively. In this paper we investigate some properties of interval cyclic colorings. In particular, we prove that if $G$ is a triangle-free graph with at least two vertices and $G \in \mathfrak{N}_c$, then $W_c(G) \leq |V(G)| + \Delta(G) - 2$. We also obtain some bounds on $w_c(G)$ and $W_c(G)$ for various classes of graphs. Finally, we give two methods for constructing of interval cyclically non-colorable graphs.



## 1. Introduction

All graphs considered in this paper are finite, undirected, and have no loops or multiple edges. Let $V(G)$ and $E(G)$ denote the sets of vertices and edges of $G$, respectively. For a graph $G$, the number of connected components of $G$ is denoted by $c(G)$. A graph $G$ is Eulerian if it has a closed trail containing every edge of $G$. The degree of a vertex $v \in V(G)$ is denoted by $d_G(v)$ (or $d(v)$), the maximum degree of $G$ by $\Delta(G)$, the diameter of $G$ by $\mathrm{diam}(G)$, and the chromatic index of $G$ by $\chi'(G)$. The terms and concepts that we do not define can be found in [1,33].

A proper edge-coloring of a graph $G$ with colors $1, \ldots, t$ is an *interval $t$-coloring* if all colors are used, and the colors of edges incident to each vertex of $G$ form an interval of integers. The concept of interval edge-coloring of graphs was introduced by Asratian and Kamalian [2]. In [2,3], the authors showed that if $G$ admits an interval coloring, then $\chi'(G) = \Delta(G)$. In [2,3], they also proved that if a triangle-free graph $G$ has an interval $t$-coloring, then $t \leq |V(G)| - 1$. Later, Kamalian [14] showed that if $G$ admits an interval $t$-coloring, then $t \leq 2|V(G)| - 3$. This upper bound was improved to $2|V(G)| - 4$ for graphs $G$ with at least three vertices [9]. For an $r$-regular graph $G$, Kamalian and Petrosyan [19] showed that if $G$ with at least $2r + 2$ vertices admits an interval $t$-coloring, then $t \leq 2|V(G)| - 5$. For a planar graph $G$, Axenovich [4] showed that if $G$ has an interval $t$-coloring, then $t \leq \frac{11}{6}|V(G)|$. In [13,14,25,28], interval colorings of complete graphs, complete bipartite graphs, trees, and $n$-dimensional cubes were investigated. The $NP$-completeness of the problem of the existence of an interval coloring of an arbitrary bipartite graph was shown in [30]. In [6,7,22,26,28,29], interval colorings of various products of graphs were investigated. In [1,3,7,8,10–12,17,18,20], the problem of the existence and construction of interval colorings was considered, and some bounds for the number of colors in such colorings of graphs were given.

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http://dx.doi.org/10.1016/j.disc.2016.01.023

A proper edge-coloring $\alpha$ of a graph $G$ with colors $1, \dots, t$ is called an *interval cyclic t-coloring* if all colors are used, and the colors of edges incident to each vertex $v$ of $G$ either form an interval of integers or the set $\{1, \dots, t\} \setminus \{\alpha(e) \colon e \text{ is incident to } v\}$ is an interval of integers. This type of edge-coloring under the name of "$\pi$-coloring" was first considered by Kotzig in [21], where he proved that every cubic graph has a $\pi$-coloring with 5 colors. However, the concept of interval cyclic edge-coloring of graphs was explicitly introduced by de Werra and Solot [5]. In [5], they proved that if $G$ is an outerplanar bipartite graph, then $G$ has an interval cyclic $t$-coloring for any $t \geq \Delta(G)$. In [23], Kubale and Nadolski showed that the problem of determining whether a given bipartite graph admits an interval cyclic coloring is *NP*-complete. Later, Nadolski [24] showed that if $G$ admits an interval coloring, then $G$ has an interval cyclic $\Delta(G)$-coloring. He also proved that if $G$ is a connected graph with $\Delta(G) = 3$, then $G$ has an interval cyclic coloring with at most 4 colors. In [15,16], Kamalian investigated interval cyclic colorings of simple cycles and trees. For simple cycles and trees, he determined all possible values of $t$ for which these graphs have an interval cyclic $t$-coloring.

In this paper we investigate some properties of interval cyclic colorings. In particular, we prove that if a triangle-free graph $G$ with at least two vertices has an interval cyclic $t$-coloring, then $t \leq |V(G)| + \Delta(G) - 2$. For various classes of graphs, we also obtain bounds on the least and the greatest values of $t$ for which these graphs have an interval cyclic $t$-coloring. Finally, we describe some methods for constructing of interval cyclically non-colorable graphs.

## 2. Notations, definitions and auxiliary results

We use standard notations $C_n, K_n$ and $Q_n$ for the simple cycle, complete graph on $n$ vertices and the hypercube, respectively. We also use standard notations $K_{m,n}$ and $K_{l,m,n}$ for the complete bipartite and tripartite graph, respectively, one part of which has $m$ vertices, other part has $n$ vertices and a third part has $l$ vertices.

A *partial edge-coloring* of $G$ is a coloring of some of the edges of $G$ such that no two adjacent edges receive the same color. If $\alpha$ is a partial edge-coloring of $G$ and $v \in V(G)$, then $S(v, \alpha)$ denotes the set of colors appearing on colored edges incident to $v$.

A graph $G$ is *interval colorable* if it has an interval $t$-coloring for some positive integer $t$. The set of all interval colorable graphs is denoted by $\mathfrak{N}$. For a graph $G \in \mathfrak{N}$, the least and the greatest values of $t$ for which it has an interval $t$-coloring are denoted by $w(G)$ and $W(G)$, respectively.

A graph $G$ is *interval cyclically colorable* if it has an interval cyclic $t$-coloring for some positive integer $t$. The set of all interval cyclically colorable graphs is denoted by $\mathfrak{N}_c$. For a graph $G \in \mathfrak{N}_c$, the least and the greatest values of $t$ for which it has an interval cyclic $t$-coloring are denoted by $w_c(G)$ and $W_c(G)$, respectively. The *feasible set* $F_c(G)$ of a graph $G$ is the set of all $t$'s such that there exists an interval cyclic $t$-coloring of $G$. The feasible set of $G$ is *gap-free* if $F_c(G) = [w_c(G), W_c(G)]$. Clearly, if $G \in \mathfrak{N}$, then $G \in \mathfrak{N}_c$ and $\chi'(G) \leq w_c(G) \leq w(G) \leq W(G) \leq W_c(G) \leq |E(G)|$.

For every positive integer $t$, let $\mathbb{Z}_t$ denote the set $\{0, \dots, t-1\}$ with addition modulo $t$ (denoted by $\oplus_t$). A norm $\|\cdot\|_t$ in $\mathbb{Z}_t$ given by $\|x\|_t = \min\{x, t-x\}$ for each $x \in \mathbb{Z}_t$. Let $\lfloor a \rfloor$ denote the largest integer less than or equal to $a$. For two positive integers $a$ and $b$ with $a \leq b$, we denote by $[a, b]$ the interval of integers $\{a, \dots, b\}$. By $[a, b]_{even}$ ($[a, b]_{odd}$), we denote the set of all even (odd) numbers from the interval $[a, b]$.

In [2,3], Asratian and Kamalian obtained the following two results.

**Theorem 1.** *If $G \in \mathfrak{N}$, then $\chi'(G) = \Delta(G)$. Moreover, if $G$ is a regular graph, then $G \in \mathfrak{N}$ if and only if $\chi'(G) = \Delta(G)$.*

**Theorem 2.** *If $G$ is a connected triangle-free graph and $G \in \mathfrak{N}$, then $W(G) \leq |V(G)| - 1$.*

For general graphs, Kamalian proved the following

**Theorem 3** (*[14]*)**.** *If $G$ is a connected graph with at least two vertices and $G \in \mathfrak{N}$, then $W(G) \leq 2|V(G)| - 3$.*

Note that the upper bound in Theorem 3 is sharp for $K_2$, but if $G \neq K_2$, then this upper bound can be improved.

**Theorem 4** (*[9]*)**.** *If $G$ is a connected graph with at least three vertices and $G \in \mathfrak{N}$, then $W(G) \leq 2|V(G)| - 4$.*

In [31], Vizing proved the following well-known result.

**Theorem 5.** *For every graph $G$,*

$$\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1.$$