Counting ordered graphs that avoid certain subgraphs

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A B S T R A C T

We say that G is an ordered graph if there is a linear ordering on the set of its vertices. In this paper we count ordered graphs that avoid a fixed forbidden ordered subgraph. We forbid each ordered graph with 2 edges and no isolated vertices. After a few trivial cases we show a coding of linear subspaces of $\mathbb{F}_2^n$ using colored M-avoiding graphs. We also consider the connection between subspaces and their orthogonal complements. We show a bijection between non-crossing and non-nesting graphs of order n and some other objects.

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1. Introduction

Extremal and enumerative problems form the two mainstream areas of combinatorics. Graph theoretical problems (both extremal and enumerative) are crucial to classical results in [1,5] and to recent research directions. Recently many classical questions were proposed while also assuming that the underlying universe has an ordered component. For example Davenport's and Schinzel's original extremal investigations of sequences were extended to matrices in [4] by Füredi and Hajnal. Permutations with forbidden patterns were enumerated extensively. Several results were obtained in [2] by Bóna and a major conjecture was proposed by R. P. Stanley and H. Wilf which was later solved in [12] by Marcus and Tardos. The two directions turned out to be strongly interlaced as proved in [7] by Klazar. Enumerating perfect matchings of [2n] with forbidden patterns led to interesting results and problems (see [11]).

A simple graph G with a linear ordering on the set of its vertices is called an *ordered* graph. In this paper we only consider ordered graphs, so we usually just write 'graph' instead of 'ordered graph'. We also assume that $V(G) = [n]$, where $[n] = \{1, \dots, n\}$. We say that G *contains* H iff there is an order-preserving mapping from $V(H)$ to $V(G)$ that maps the edges of H into edges of G. We say that G *avoids* H if G does not contain H. The *size* of a graph H is the number of its edges and the *order* of H is the number of its vertices. We denote the set of H-avoiding graphs on $[n]$ by $A_n(H)$.

By fixing H one can consider the set of ordered graphs with certain size or order that avoid H. Extremal and enumerative questions about these sets have already been studied in the last few years. Many asymptotic results can be found in [10] by Klazar. Pach and Tardos published about extremal results in [13].

Our goal is to determine the number of graphs that avoids a fixed subgraph H of a certain order. We choose the forbidden graph H as each of the 6 ordered graphs with 2 edges that contain no isolated vertices.

In Section 2 we consider some simple cases for the sake of completeness. In Section 3 we work with the M graph (see Fig. 1 where all graphs considered in this paper are listed). We establish two bijections between certain vertex-colored M-avoiding graphs and linear subspaces of $\mathbb{F}_2^n$. These bijections can be interpreted as codings of linear subspaces of $\mathbb{F}_2^n$. These codings reflect some linear algebraic properties of the subspace. For example we show how these codings behave with orthogonal subspaces.
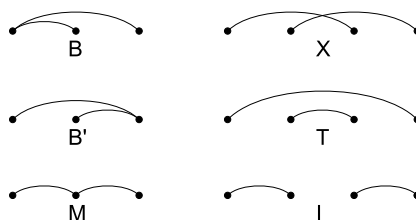
Fig. 1. The set of ordered graphs containing 2 edges and no isolated vertices.

In Section 4 we show a size and order-preserving bijection between T-avoiding (non-nesting) and X-avoiding (non-intersecting) graphs. It has already been known that non-nesting and non-intersecting graphs of the same order/size have the same number (see [6,8,9]), but no bijective proof has been published so far. After the main result we extend the bijections between non-nesting and non-crossing graphs to other families.

2. Warmup

Let $\{u, v\} \in E(G)$ be an arbitrary edge. By $(u, v) \in E(G)$ we mean that $\{u, v\} \in G$ and $u < v$.

Let us consider the graph B first. We say that v is the *right neighbor* of u iff $(u, v) \in E(G)$. This is equivalent to u being the *left neighbor* of v. It is easy to see that G avoids B iff all vertices in G have at most 1 right neighbor. For vertex u this gives $n - u + 1$ ways to choose right neighbors, and all these choices are independent. This means that there are exactly $n!$ such graphs. Similarly it can be shown that there are also $n!$ graphs on $[n]$ that avoid B' by switching the words 'left' and 'right'.

A less trivial case to consider is avoiding the I graph.

It is easy to see that G avoids I iff for every (u, v), $(u', v') \in E(G)$ the intervals $[u, v]$ and $[u', v']$ intersect. Using Helly's theorem this means that $\bigcap_{(u,v) \in E(G)} [u, v] \neq \emptyset$. Define $Int(G) := \bigcap_{(u,v) \in E(G)} [u, v]$.

Proposition 1. *For all $n \geq 0$, $|A_n(I)| = 1 + \sum_{i=1}^{n-1} (2^{n-i} - 1) \cdot 2^{(i-1) \cdot (n-i+1)}$.*

Proof. Let $G \in A_n(I)$ be arbitrary. If there are no edges in G then G is the empty graph, so there is only one such G.

If G has at least one edge then $i := \min Int(G)$. It is trivial that $1 \leq i \leq n - 1$ always holds. There exists $v \in V(G)$ for which $(i, v) \in E(G)$. It also holds that for all $(u', v') \in E(G) : u' \leq i \leq v'$. There are $2^{n-i} - 1$ ways to choose the right neighborhood of i, and $2^{(i-1) \cdot (n-i+1)}$ ways to choose edges which do not contain i as their left vertex. We can choose these independently. Summing over i and counting the empty graph gives us the result. $\square$

3. The M-graph

Our next subject of study is the M-graph. Note that G avoids M iff there is no vertex which has both a left-neighbor and a right-neighbor. By $\langle x_1, x_2, \ldots, x_n \rangle$ we mean the set $\{x_1, x_2, \ldots, x_n\}$ while also assuming $x_1 < x_2 < \cdots < x_n$.

Proposition 2. *M-avoiding graphs are bipartite.*

Proof. Let $G \in A_n(M)$ be arbitrary, and let R be the set of vertices in M which have a right neighbor. Let L be $V(G) \setminus R$, the set of vertices that are either isolated or have left neighbors. Let $(i, j) \in E(G)$ be an arbitrary edge. Since i has a right neighbor j, we know that $i \in R$. If $j \in R$, then j has both a left and a right neighbor too, which would be a contradiction. So $j \in L$. This partition $\{R, L\}$ of $V(G)$ shows that G is bipartite. $\square$

We say that (G, c) is a *nicely colored M-avoiding graph* if $G \in A_n(M)$ and $c : V(G) \to \{blue, red\}$ is a proper 2-coloring, which colors every vertex with a right neighbor *blue*. Note that this means that each vertex with a left neighbor is colored *red*. Isolated vertices can get either color, independently. This means that for each M-avoiding graph G there are 2^m choices of c to get a nice coloring of G, where m is the number of isolated vertices. Let us denote the set of nicely colored M-avoiding graphs on $[n]$ by B_n.

Proposition 3. *For all $n \geq 0$, $|B_n| = \sum_{k=0}^{n} \binom{n}{k} |A_k(M)|$.*

Proof. Choosing a nicely colored M-avoiding graph is equivalent to choosing a subset U of $[n]$ (which contains the non-isolated vertices and *blue* isolated vertices), and choosing an M-avoiding ordered graph G' on U. There are $\binom{n}{k}$ ways to choose U, assuming U is a k-subset. For each choice there are $|A_k(M)|$ ways to choose an M-avoiding ordered graph on U. We can get a nicely colored M-avoiding graph by coloring each isolated vertex *blue* if it is contained in U, *red* otherwise. Non-isolated vertices get *blue* color if they have right neighbors only, *red* otherwise. It is easy to see that when given a nicely colored M-avoiding graph on $[n]$, we can uniquely determine G' and U.

The proposition follows by directly enumerating the ways to choose U and G'. $\square$