Discrete Mathematics 339 (2016) 1878–1883

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Independence in uniform linear triangle-free hypergraphs

Piotr Borowiecki [a], Michael Gentner [b], Christian Löwenstein [b], Dieter Rautenbach [b,*]

[a] *Department of Algorithms and System Modeling, Faculty of Electronics, Telecommunications and Informatics, Gdańsk University of Technology, 80-233 Gdańsk, Poland*

[b] *Institute of Optimization and Operations Research, Ulm University, D-89069 Ulm, Germany*

ARTICLE INFO

Article history:
Received 2 October 2014
Accepted 5 January 2016
Available online 1 February 2016

Keywords:
Independence
Hypergraph
Linear
Uniform
Double linear
Triangle-free

ABSTRACT

The independence number $\alpha(H)$ of a hypergraph H is the maximum cardinality of a set of vertices of H that does not contain an edge of H. Generalizing Shearer's classical lower bound on the independence number of triangle-free graphs Shearer (1991), and considerably improving recent results of Li and Zang (2006) and Chishti et al. (2014), we show that

$$\alpha(H) \geq \sum_{u \in V(H)} f_r(d_H(u))$$

for an r-uniform linear triangle-free hypergraph H with $r \geq 2$, where

$$f_r(0) = 1, \quad \text{and}$$
$$f_r(d) = \frac{1 + \big((r-1)d^2 - d\big)f_r(d-1)}{1 + (r-1)d^2} \quad \text{for } d \geq 1.$$

1. Introduction

We consider finite *hypergraphs* H, which are ordered pairs $(V(H), E(H))$ of two sets, where $V(H)$ is the finite set of *vertices of* H and $E(H)$ is the *set of edges of* H, which are subsets of $V(H)$. The *order* $n(H)$ *of* H is the cardinality of $V(H)$. The *degree* $d_H(u)$ *of a vertex* u *of* H is the number of edges of H that contain u. The *average degree* $d(H)$ *of* H is the arithmetic mean of the degrees of its vertices. Two distinct vertices of H are *adjacent* or *neighbors* if some edge of H contains both. The *neighborhood* $N_H(u)$ *of a vertex* u *of* H is the set of vertices of H that are adjacent to u. For a set X of vertices of H, the hypergraph $H - X$ arises from H by removing from $V(H)$ all vertices in X and removing from $E(H)$ all edges that intersect X. If every two distinct edges of H share at most one vertex, then H is *linear*. If H is linear and for every two distinct non-adjacent vertices u and v of H, every edge of H that contains u contains at most one neighbor of v, then H is *double linear*. If there are not three distinct vertices u_1, u_2, and u_3 of H and three distinct edges e_1, e_2, and e_3 of H such that $\{u_1, u_2, u_3\} \setminus \{u_i\} \subseteq e_i$ for $i \in \{1, 2, 3\}$, then H is *triangle-free*. A set I of vertices of H is a *(weak) independent set of* H if no edge of H is contained in I. The *(weak) independence number* $\alpha(H)$ *of* H is the maximum cardinality of an independent set of H. If all edges of H have cardinality r, then H is r-*uniform*. If H is 2-uniform, then H is referred to as a *graph*.

* Corresponding author.
E-mail addresses: pborowie@eti.pg.gda.pl (P. Borowiecki), michael.gentner@uni-ulm.de (M. Gentner), christian.loewenstein@uni-ulm.de (C. Löwenstein), dieter.rautenbach@uni-ulm.de (D. Rautenbach).

http://dx.doi.org/10.1016/j.disc.2016.01.006

The independence number of (hyper)graphs is a well studied computationally hard parameter. Caro [4] and Wei [14] proved a classical lower bound on the independence number of graphs, which was extended to hypergraphs by Caro and Tuza [5]. Specifically, for an r-uniform hypergraph H, Caro and Tuza [5] proved

$$\alpha(H) \geq \sum_{u \in V(H)} f_{CT(r)}(d_H(u)),$$

where $f_{CT(r)}(d) = \binom{d+\frac{1}{r-1}}{d}^{-1}$. Thiele [13] generalized Caro and Tuza's bound to general hypergraphs; see [3] for a very simple probabilistic proof of Thiele's bound. Originally motivated by Ramsey theory, Ajtai et al. [2] showed that $\alpha(G) = \Omega\left(\frac{\ln d(G)}{d(G)} n(G)\right)$ for every triangle-free graph G. Confirming a conjecture from [2] concerning the implicit constant, Shearer [11] improved this bound to $\alpha(H) \geq f_{S_1}(d(G))n(G)$, where $f_{S_1}(d) = \frac{d \ln d - d + 1}{(d-1)^2}$. In [11] the function f_{S_1} arises as a solution of the differential equation

$$(d+1)f(d) = 1 + (d - d^2)f'(d) \quad \text{and} \quad f(0) = 1.$$

In [12] Shearer showed that

$$\alpha(G) \geq \sum_{u \in V(G)} f_{S_2}(d_G(u))$$

for every triangle-free graph G, where f_{S_2} solves the difference equation

$$(d+1)f(d) = 1 + (d - d^2)\Big(f(d) - f(d-1)\Big) \quad \text{and} \quad f(0) = 1.$$

Since $f_{S_1}(d) \leq f_{S_2}(d)$ for every non-negative integer d, and f_{S_1} is convex, Shearer's bound from [12] is stronger than his bound from [11].

Li and Zang [10] adapted Shearer's approach to hypergraphs and obtained the following.

Theorem 1 (*Li and Zang [10]*). *Let r and m be positive integers with $r \geq 2$.*

If H is an r-uniform double linear hypergraph such that the maximum degree of every subhypergraph of H induced by the neighborhood of a vertex of H is less than m, then

$$\alpha(H) \geq \sum_{u \in V(H)} f_{LZ(r,m)}(d_H(u)),$$

where

$$f_{LZ(r,m)}(x) = \frac{m}{B} \int_0^1 \frac{(1-t)^{\frac{a}{m}}}{t^b(m-(x-m)t)} dt,$$

$a = \frac{1}{(r-1)^2}$, $b = \frac{r-2}{r-1}$, and $B = \int_0^1 (1-t)^{(\frac{a}{m}-1)} t^{-b} dt$.

Note that for $r \geq 2$, an r-uniform linear hypergraph H is triangle-free if and only if it is double linear and the maximum degree of every subhypergraph of H induced by the neighborhood of a vertex of H is less than 1. Therefore, since $f_{S_1} = f_{LZ(2,1)}$ and f_{S_1} is convex, Theorem 1 implies Shearer's bound from [11]. Nevertheless, since $f_{S_1}(d) < f_{S_2}(d)$ for every integer d with $d \geq 2$, Shearer's bound from [12] does not quite follow from Theorem 1.

In [6] Chishti et al. presented another version of Shearer's bound from [11] for hypergraphs.

Theorem 2 (*Chishti et al. [6]*). *Let r be an integer with $r \geq 2$.*

If H is an r-uniform linear triangle-free hypergraph, then

$$\alpha(H) \geq f_{CZPI(r)}(d(H))n(H),$$

where

$$f_{CZPI(r)}(x) = \frac{1}{r-1} \int_0^1 \frac{1-t}{t^b(1-((r-1)x-1)t)} dt$$

and $b = \frac{r-2}{r-1}$.

Since $f_{S_1} = f_{CZPI(2)}$, for $r = 2$, the last result coincides with Shearer's bound from [11].

A drawback of the bounds in Theorems 1 and 2 is that they are very often weaker than Caro and Tuza's bound [5], which holds for a more general class of hypergraphs. See Fig. 1 for an illustration.