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Domination stability in graphs

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ABSTRACT

For a graph $G = (V, E)$, a subset $D \subseteq V(G)$ is a dominating set if every vertex of $V(G) \setminus D$ has a neighbor in D. The domination number of G is the minimum cardinality of a dominating set of G. The domination stability, or just γ-stability, of a graph G is the minimum number of vertices whose removal changes the domination number. We show that the γ-stability problem is NP-hard even when restricted to bipartite graphs. We obtain several bounds, exact values and characterizations for the γ-stability of a graph, and we characterize the trees with $st_\gamma(T) = 2$.

1. Introduction

Let G be a graph. A subset $D \subseteq V(G)$ is a dominating set of G if every vertex of $V(G) \setminus D$ has a neighbor in D. The domination number of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. A dominating set of G of minimum cardinality is called a $\gamma(G)$-set. For a comprehensive survey of domination in graphs, see [7].

A domination-critical vertex in a graph G is a vertex whose removal decreases the domination number. One of important problems in domination theory is to determine graphs in which every vertex is critical, see for example [1,2,6,9,10]. Much have been also written about graphs with no critical vertex, see [3,4,8].

Bauer et al. [1] introduced the concept of domination stability in graphs. The domination stability, or just γ-stability, of a graph G is the minimum number of vertices whose removal changes the domination number. The γ^--stability of G, denoted by $\gamma^-(G)$, is defined as the minimum number of vertices whose removal decreases the domination number, and the γ^+-stability of G, denoted by $\gamma^+(G)$, is defined as the minimum number of vertices whose removal increases the domination number. We denote the γ-stability of G by $st_\gamma(G)$. Thus the domination stability of a graph G is $st_\gamma(G) = \min\{\gamma^-(G), \gamma^+(G)\}$.

The open neighborhood of a vertex v of G is the set $N_G(v) = \{u \in V(G): uv \in E(G)\}$. The closed neighborhood of v is $N_G[v] = N_G(v) \cup \{v\}$. For a subset $S \subseteq V(G)$, we define $N_G(S) = \cup_{v\in S} N_G(v)$ and $N_G[S] = \cup_{v\in S} N_G[v]$. The private neighborhood of a vertex $v \in S$ is $\mathrm{pn}_G(v, S) = \{u \in V(G): N_G(u) \cap S = \{v\}\}$. Each vertex in $\mathrm{pn}_G(v, S)$ is called a private neighbor of v. The external private neighborhood $\mathrm{epn}(v, S)$ of v with respect to S consists of those private neighbors of v in $V(G) \setminus S$. Thus $\mathrm{epn}(v, S) = \mathrm{pn}(v, S) \setminus S$. The degree of a vertex v, that is, the cardinality of its open neighborhood, is denoted by $d_G(v)$. By a leaf we mean a vertex of degree one, while a support vertex is a vertex adjacent to a leaf. We say that a support vertex is strong (weak, respectively) if it is adjacent to at least two leaves (exactly one leaf, respectively). The maximum (minimum, respectively) degree among all vertices of G is denoted by $\Delta(G)$ ($\delta(G)$, respectively). The distance between two vertices of a graph is the number of edges in a shortest path connecting them. The eccentricity of a vertex is the greatest distance between it and any other vertex. The diameter of a graph G, denoted by $\mathrm{diam}(G)$, is the maximum

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eccentricity among all vertices of G. The complete graph on n vertices we denote by K_n. The path (cycle, respectively) on n vertices we denote by P_n (C_n, respectively). Let T be a tree, and let v be a vertex of T. We say that v is adjacent to a path P_n if there is a neighbor of v, say x, such that a subtree resulting from T by removing the edge vx is a path P_n in which the vertex x is a leaf. By a star we mean a connected graph in which exactly one vertex has degree greater than one. Double star is a graph obtained from a star by joining a positive number of vertices to one of the leaves. Let uv be an edge of a graph G. By subdividing the edge uv we mean removing it, and adding a new vertex, say x, along with two new edges ux and xv. By contracting the edge uv we mean replacing uv and the vertices u and v with a new vertex adjacent to all neighbors of u or v in G. If S is a subset of $V(G)$, then we denote by $G[S]$ the subgraph of G induced by the vertices of S.

It can be easily seen that if G is a disconnected graph with components $G_1, G_2, \ldots, G_k$, then $st_\gamma(G) = \min\{st_\gamma(G_1), st_\gamma(G_2), \ldots, st_\gamma(G_k)\}$. Hence we only study connected graphs.

For a graph G, let $\rho(G) = \min\{|\mathrm{epn}_G(v, S)| : v \in S, S \text{ is a } \gamma(G)\text{-set}\}$.

Bauer et al. [1] obtained the following necessary and sufficient condition for a graph to have a domination-critical vertex.

Proposition 1 ([1]). *A graph G has a domination-critical vertex if and only if $\rho(G) = 0$.*

The following upper bound is known for the γ-stability of any graph.

Proposition 2 ([1]). *For every graph G we have $st_\gamma(G) \le \delta(G) + 1$.*

We show that the γ-stability problem is NP-hard even when restricted to bipartite graphs. We obtain several bounds, exact values and characterizations for the γ-stability of a graph, and we characterize the trees with $st_\gamma(T) = 2$.

2. Complexity

This section concerns the NP-hardness of the γ-stability decision problem.

DOMINATION STABILITY PROBLEM

INSTANCE: A graph $G = (V, E)$ and the domination number $\gamma(G)$.

QUESTION: Is $st_\gamma(G) > 1$?

Dettlaff et al. [5] studied the complexity of determining domination subdivision numbers of graphs. The domination subdivision number $sd(G)$ of a graph G is the minimum number of edges in G that must be subdivided (where an edge can be subdivided only once) in order to increase the domination number. Dettlaff et al. proved that the decision problem for domination subdivision number is NP-hard even for bipartite graphs (see Theorem 1 of [5]). Their proof was performed by a transformation from 3-SAT and usage of a gadget. With a similar proof using the same gadget and a transformation from 3-SAT, we can obtain the following result.

Theorem 3. *The domination stability problem is NP-hard even for bipartite graphs.*

Since the class of graphs with $st_\gamma(G) > 1$ is a subclass of graphs with no domination-critical vertex, we have the following result.

Theorem 4. *The decision problem for determining graphs with no domination-critical vertex is NP-hard even for bipartite graphs.*

3. Exact values

In this section we determine the domination stability for some classes of graphs.

Observation 5. *If G is a star or a double star, then $st_\gamma(G) = 1$.*

Observation 6. *For complete bipartite graphs $K_{m,n}$ with $2 \le m \le n$ we have $st_\gamma(K_{m,n}) = m - 1$.*

Observation 7. *We have $\gamma(P_n) = \gamma(C_n) = \lfloor (n+2)/3 \rfloor$.*

First we investigate the γ-stability of paths.

Proposition 8. *For paths P_n we have $st_\gamma(P_n) = 2$ if $n \equiv 2 \pmod 3$, and $st_\gamma(P_n) = 1$ otherwise.*

Proof. First assume that $n \equiv 0 \pmod 3$. Let us observe that $\gamma(P_n - v) = \gamma(P_n) + 1$, where v is a support vertex. Consequently, $st_\gamma(P_n) = 1$. Next assume that $n \equiv 1 \pmod 3$. If v is a leaf, then $\gamma(P_n - v) = \gamma(P_n) - 1$, and consequently, $st_\gamma(P_n) = 1$. Now assume that $n = 3k + 2$ for some integer k. Using Observation 7 we get $\gamma(P_n) = k + 1$. Let v be an arbitrary vertex of P_n. We show that the removal of v does not change the domination number. If v is a leaf, then $\gamma(P_n - v) = \gamma(P_{n-1}) = k + 1 = \gamma(P_n)$. Now assume that the degree of v is 2. Let P_{n_1} and P_{n_2} be the components of $P_n - v$. Without loss of generality we may assume that either $n_1 \equiv 0 \pmod 3$ and $n_2 \equiv 1 \pmod 3$, or $n_1 \equiv 2 \pmod 3$ and $n_2 \equiv 2 \pmod 3$. In the first case we get $\gamma(P_n - v) = \gamma(P_{n_1}) + \gamma(P_{n_2}) = \lfloor (n_1 + 2)/3 \rfloor + \lfloor (n_2 + 2)/3 \rfloor = n_1/3 + (n_2 + 2)/3 = (n + 1)/3 = k + 1 = \gamma(P_n)$. In the second case we similarly obtain $\gamma(P_n - v) = \gamma(P_n)$. We conclude that $st_\gamma(P_n) \ge 2$. Now, Proposition 2 implies that $st_\gamma(P_n) = 2$. ■