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Spectra of graphs and closed distance magic labelings

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ABSTRACT

Let $G = (V, E)$ be a graph of order n. A closed distance magic labeling of G is a bijection $\ell : V(G) \to \{1, \dots, n\}$ for which there exists a positive integer k such that $\sum_{x \in N[v]} \ell(x) = k$ for all $v \in V$, where $N[v]$ is the closed neighborhood of v. We consider the closed distance magic graphs in the algebraic context. In particular we analyze the relations between the closed distance magic labelings and the spectra of graphs. These results are then applied to the strong product of graphs with complete graph or cycle and to the circulant graphs. We end with a number theoretic problem whose solution results in another family of closed distance magic graphs somewhat related to the strong product.

1. Introduction and preliminaries

All graphs considered in this paper are simple finite graphs. For a graph G, we use $V(G)$ for the vertex set and $E(G)$ for the edge set of G. The *open neighborhood* $N(x)$ (or more precisely $N_G(x)$, when needed) of a vertex x is the set of all vertices adjacent to x, and the *degree* $d(x)$ of x is $|N(x)|$, i.e. the size of the neighborhood of x. By $N[x]$ (or $N_G[x]$) we denote the *closed neighborhood* $N(x) \cup \{x\}$ of x. By C_n we denote a cycle on n vertices.

Different kinds of labelings have been important part of graph theory. See a dynamic survey [10] which covers the field. One type of labelings includes magic labelings, where some elements (edges, vertices, etc.) of a graph must be labeled in such a way that certain sums (depending on graph properties) are constant. *Closed distance magic labeling* (also called *Σ'-labeling*, see [3]) of a graph $G = (V(G), E(G))$ of order n is a bijection $\ell : V(G) \to \{1, \dots, n\}$ with the property that there is a positive integer k' (called the *magic constant*) such that $w(x) = \sum_{y \in N_G[x]} \ell(y) = k'$ for every $x \in V(G)$, where $w(x)$ is the *weight* of x. If a graph G admits a closed distance magic labeling, then we say that G is *closed distance magic graph*. Closed distance magic graphs are an analogue to distance magic graphs, where the sums are taken over the open neighborhoods $N_G(x)$ instead of the closed ones $N_G[x]$, see [2,7,8].

Let D be a subset of non-negative integers. O'Neal and Slater in [16] have defined the D-distance magic labeling as a bijection $f : V(G) \to \{1, \dots, n\}$ such that there is a magic constant k such that for any vertex $x \in V(G)$, $w(x) = \sum_{y \in N_D(x)} f(y) = k$. Here $N_D(x) = \{y \in V(G) | d(x, y) \in D\}$, i.e., the weight of a vertex $x \in V(G)$ is the sum of the labels of all the vertices $y \in V(G)$ for which their distance to x belongs to D. This kind of labeling has been studied e.g. by Simanjuntak et al. in [18] (we refer to some of their results in one of the following sections). It is a generalization of both distance magic labeling and closed distance magic labeling, where $D = \{1\}$ and $D = \{0, 1\}$, respectively.

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The concept of distance magic labeling has been motivated by the construction of magic rectangles. Magic rectangles are natural generalization of magic squares that has been intriguing mathematicians and the general public for a long time [11]. A magic (m, n)-rectangle S is an $m \times n$ array in which the first mn positive integers are placed so that the sum over each row of S is constant and the sum over each column of S is another (different if $m \neq n$) constant. Harmuth proved that:

Theorem 1.1 ([13,14]). *For $m, n > 1$ there is a magic (m, n)-rectangle S if and only if $m \equiv n \pmod 2$ and $(m, n) \neq (2, 2)$.*

A related concept is the notion of *distance antimagic labeling*. This is again a bijection $\overline{f}$ from $V(G)$ to $\{1, \ldots, n\}$ but this time different vertices are required to have distinct weights (where the sums are taken over the open neighborhoods $N_G(x)$). A more restrictive version of this labeling is the (a, d)-distance antimagic labeling. It is a distance antimagic labeling with the additional property that the weights of vertices form an arithmetic progression with difference d and first term a. If $d = 1$, then $\overline{f}$ is called simply *distance antimagic labeling* [9]. Notice that if a graph has a closed distance magic labeling then it has a distance antimagic labeling. The opposite is however not true, as it can be easily checked on the example of C_5. It has no closed distance magic labeling, while it has a distance antimagic labeling $\overline{f}(v_i) = i$, where $V(C_5) = \{v_1, v_2, v_3, v_4, v_5\}$ and $E(C_5) = \{\{v_i, v_j\} : |i - j| = 1 \vee |i - j| = 4\}$.

Finding an r-regular distance antimagic labeling turns out to be equivalent to finding a fair incomplete tournament FIT(n, r) [9]. A *fair incomplete tournament* of n teams with g rounds, FIT(n, r), is a tournament in which every team plays r other teams and the total strength of the opponents that team i plays is $S_{n,r}(i) = (n + 1)(n - 2)/2 + i - c$ for every i and some fixed constant c.

We recall one of four standard graph products (see [12]). Let G and H be two graphs. The *strong product* $G \boxtimes H$ is a graph with vertex set $V(G) \times V(H)$. Two vertices (g, h) and (g', h') are adjacent in $G \boxtimes H$ if either $g = g'$ and h is adjacent with h' in H, or $h = h'$ and g is adjacent with g' in G, or g is adjacent with g' in G and h is adjacent with h' in H. Recently in [1] two other standard products, namely direct and lexicographic, have been considered with respect to the property of being distance magic.

It is easy to notice the following observation, that will be useful in our further considerations.

Observation 1.2. *If G is an r-regular closed distance magic graph on n vertices, then $k' = \frac{(r+1)(n+1)}{2}$.*

In the next section we reveal somewhat surprising connection between the existence of closed distance magic labeling and the spectrum of a graph. In the following sections we consider the existence of closed distance magic labelings of chosen families of graphs using algebraic tools developed in Section 2. In the final section we present a combinatorial problem whose solution yields more closed distance magic graphs.

2. Necessary conditions—algebraic approach

Let $V(G) = \{x_1, \ldots, x_n\}$. Then the following system of equations with unknowns $\ell(x_1), \ldots, \ell(x_n)$ has to be satisfied for every closed distance magic graph:

$$\begin{aligned} w(x_1) &= k', \\ w(x_2) &= k', \\ &\vdots \\ w(x_n) &= k'. \end{aligned} \tag{1}$$

By writing this system in matrix form, we get

$$(\mathbf{A}(G) + \mathbf{I}_n)\mathbf{l} = k'\mathbf{u}_n,$$

where $\mathbf{A}(G)$ is the adjacency matrix of G, $\mathbf{I}_n$ is $n \times n$ identity matrix, $\mathbf{l} = (\ell(x_1), \ldots, \ell(x_n))$ and $\mathbf{u}_n$ is a vector of length n with every entry equal to 1.

It is well known that the rank of a square matrix is equal to the number of its non-zero singular values (see e.g. [6], p. 31). In the case of symmetric matrices it is in turn equal to the number of the non-zero eigenvalues. It means that the dimension of the set of solutions of the system (1) equals to the multiplicity of 0 in the spectrum of $\mathbf{A}(G) + \mathbf{I}_n$, i.e., to the multiplicity of -1 in $Sp(G)$, where $Sp(G)$ denotes the spectrum of G.

It has been recently proved by O'Neil and Slater in [17] that if a graph is closed distance magic, then the magic constant k' is unique, i.e., even if there exist two distinct closed distance magic labelings, then they result in same magic constant k'.

The above considerations lead us to the following result.

Theorem 2.1. *If G is a closed distance magic graph and the system (1) has $k + 1$ linearly independent solutions, then the multiplicity of -1 in $Sp(G)$ is k.*

The following corollary will be used in the remainder of the paper.

Corollary 2.2. *Let G be a closed distance magic graph such that there exist k linearly independent solutions to the system (1) such that no bijection $\ell : V(G) \to \{1, \ldots, |V(G)|\}$ is their linear combination. Then the multiplicity of -1 in $Sp(G)$ is at least k.*