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Anti-Ramsey numbers in complete split graphs

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ABSTRACT

A subgraph of an edge-colored graph is *rainbow* if all of its edges have different colors. The *anti-Ramsey number* $ar(G, H)$ is the maximum number of colors in an edge-coloring of G with no rainbow copy of H. Anti-Ramsey numbers were introduced by Erdős et al. (1973) and studied in numerous papers. Originally a complete graph was considered as G, but afterwards also other graphs were used as host graphs.

We consider a complete split graph as the host graph and discuss some results for the graph H containing short cycles or triangles with pendant edges. Among others we show that $ar(K_n + \overline{K_s}, C_3^+) = ar(K_n + \overline{K_s}, C_3) = n + s - 1$ for $n, s \geq 1$, where C_3^+ denotes a triangle with a pendant edge.

1. Introduction

A subgraph of an edge-colored graph is *rainbow* if all of its edges have different colors. For graphs G and H *the anti-Ramsey number* $ar(G, H)$ is the maximum number of colors in an edge-coloring of G with no rainbow copy of H. Anti-Ramsey numbers were introduced by Erdős et al. [6] and considered in the classical case when $G = K_n$. Since then numerous results were established for a variety of graphs H, including among others cycles [1,15,18], matchings [7,11,19] and trees [14,16]. Later on different graphs were considered as a graph G, for instance bipartite graphs [3,17] or hypercubes [2]. The paper of Fujita, Magnant and Ozeki [8] presents a survey of results in classical and nonclassical case. Very recently a set of triangulations was placed as G [12,13].

In the paper we consider complete split graphs as the graph G and discuss some results concerning short cycles and triangles with pendant edges. Among others we show that $ar(K_n + \overline{K_s}, C_3^+) = ar(K_n + \overline{K_s}, C_3) = n + s - 1$ for $n, s \geq 1$, where C_3^+ denotes a triangle with a pendant edge.

2. Preliminaries

Graphs considered below will always be simple. Throughout the paper we use the standard graph theory notation (see, e.g., [5]). In particular, $G \cup H$, K_n, C_n, P_n and $K_{1,r}$ stand, respectively, for disjoint sum of graphs G and H, the complete graph, the cycle, the path on n vertices and a star with r rays. A symbol $K_{1,r} + e$ denotes a star with one edge added. An edge of a graph is called *pendant* if exactly one of its ends has the degree 1. For a graph G and its subgraph H by $G - H$ we mean a graph obtained from G by deleting all vertices of H. For a set S by $|S|$ we denote the cardinality of S.

Additionally we introduce the following notation. $C(G)$ is a set of colors used on the edges of a graph G; $C(v)$ is a set of colors used on the edges incident to a vertex v and $c(e)$ denotes the color of the edge e.

A complete split graph $K_n + \overline{K_s}$ is a join of a complete graph K_n and an empty graph $\overline{K_s}$. Throughout the paper $V(K_n) = \{x_1, x_2, \ldots, x_n\} = X$ and $V(\overline{K_s}) = \{y_1, y_2, \ldots, y_s\} = Y$. We will also be referring to a bipartite graph $K_{n,s}$.

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We will also need the following theorem by Erdős, Simonovits, Sós.

Theorem 1 ([6]). *Let $n \geq 3$. Then $ar(K_n, C_3) = n - 1$.*

3. General observations

We start with some results which are not difficult to show but, due to their generality, can be used in more particular cases.

Theorem 2. *Let H be a graph with $\delta(H) \geq 2$. Then $ar(K_n + \overline{K_s}, H) \geq ar(K_n, H) + s$, for $n, s \geq 1$.*

Proof. To show this bound it is enough to color the edges of K_n from $K_n + \overline{K_s}$ with $ar(K_n, H)$ colors avoiding rainbow H, and then add s monochromatic stars (each in a new color) coming out from s vertices of $\overline{K_s}$. □

As it will be shown later on this lower bound is sharp for instance for a triangle (see Corollary 3).

Theorem 3. *If $|V(H)| \leq n$ and H is a subgraph of $K_{n,s}$ then $ar(K_n + \overline{K_s}, H) \leq ar(K_n, H) + ar(K_{n,s}, H)$.*

Proof. Consider an arbitrary coloring of the edges of $K_n + \overline{K_s}$ with $ar(K_n, H) + ar(K_{n,s}, H) + 1$ colors. At least one of the following inequalities is true $|C(K_n)| \geq ar(K_n, H) + 1$ or $|C(K_{n,s})| \geq ar(K_{n,s}, H) + 1$ so we obtain a rainbow copy of H by the definition of the anti-Ramsey number. □

The sharpness of this bound will be discussed in the next paragraph for a 4-cycle.

4. Cycles

Here we will present some results which we are able to derive for anti-Ramsey numbers for short cycles.

Corollary 1. *Let $3 \leq t \leq \min\{n + s, 2n\} = p$. Then*

$$ar(K_n + \overline{K_s}, C_t) \geq \begin{cases} ar(K_n, C_t) + s, & 3 \leq t \leq n \\ \binom{n}{2} + (n-1)(t-n-1) + s, & n+1 \leq t \leq p. \end{cases}$$

Proof. The first inequality is straightforward from Theorem 2. As for the second one we color the edges of K_n rainbowly, add $n(t - 1 - n)$ new colors on the edges of $K_{n,t-1-n}$ and finally add $s - (t - 1 - n)$ monochromatic stars, each in a new color. □

We start with a triangle. The lower bound presented below is straightforward from Corollary 1 and Theorem 1. The upper one is a consequence of the upper bound for a triangle with a pendant edge which is presented in Theorem 11 and occurs to be the same.

Theorem 4. *Let $n \geq 2$, $s \geq 1$ and $n + s \geq 3$. Then $ar(K_n + \overline{K_s}, C_3) = n + s - 1$.*

Next we consider a 4-cycle. It is the graph which fulfills the assumptions of Theorem 3 and that is why it seems to be interesting since it can occur in a complete part of a complete split graph, in its bipartite part or somewhere in between. The cases where the host graph is complete or complete bipartite are completely solved.

Theorem 5 ([1]). *Let $n \geq 4$. Then $ar(K_n, C_4) = \lfloor \frac{4n}{3} \rfloor - 1$.*

Theorem 6 ([4]). *Let $n \leq s$ and $k \geq 2$. Then*

$$ar(K_{n,s}, C_{2k}) = \begin{cases} (k-1)(n+s) - 2(k-1)^2 + 1 & \text{for } n \geq 2k-1 \\ (k-1)s + n - (k-1) & \text{for } k-1 \leq n \leq 2k-1 \\ sn & \text{for } n \leq k-1. \end{cases}$$

Firstly we consider cases in which C_4 is not a subgraph of a complete part.

Proposition 1. *Let $n \in \{2, 3\}$ and $n + s \geq 4$. Then $ar(K_n + \overline{K_s}, C_4) = ar(K_{n,s}, C_4) + 1$.*

Proof. It is easy to see that the edge of K_2 has no influence of appearing of (rainbow) C_4 in $K_2 + \overline{K_s}$ so it can be colored with an additional color. Similarly in case $n = 3$ we can color the edges of $K_{3,s}$ with $ar(K_{n,s}, C_4)$ colors without rainbow C_4 and the edges of a triangle with one additional color to avoid a rainbow C_4. On the other hand if we color the edges of $K_n + \overline{K_s}$ with at least $ar(K_{n,s}, C_4) + 2$ then at least two of them are used on the edges of K_n otherwise we have a rainbow C_4 in $K_{n,s}$. It is impossible in case $n = 2$. In case $n = 3$ note that there is at least one vertex $y_i \in Y$ such that $|C(y_i) \setminus C(K_3)| \geq 2$. If this rainbow path P_3 has common leaves with a rainbow path P_3 in K_3 then we are done. If not, no matter of which color is the edge $y_i x_k$, where x_k is the center of the monochromatic path in K_3 we always obtain a rainbow C_4. □

Next we show that if the assumptions of Theorem 3 are fulfilled then in case of C_4 the upper bound can be decreased by 1.