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Matchings, path covers and domination

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We show that if G is a graph with minimum degree at least three, then $\gamma_t(G) \le \alpha'(G) + (\mathrm{pc}(G)-1)/2$ and this bound is tight, where $\gamma_t(G)$ is the total domination number of G, $\alpha'(G)$ the matching number of G and $\mathrm{pc}(G)$ the path covering number of G which is the minimum number of vertex disjoint paths such that every vertex belongs to a path in the cover. We show that if G is a connected graph on at least six vertices, then $\gamma_{\mathrm{nt}}(G) \le \alpha'(G) + \mathrm{pc}(G)/2$ and this bound is tight, where $\gamma_{\mathrm{nt}}(G)$ denotes the neighborhood total domination number of G. We observe that every graph G of order n satisfies $\alpha'(G) + \mathrm{pc}(G)/2 \ge n/2$, and we characterize the trees achieving equality in this bound.

1. Introduction

Given a graph G, a set $S \subseteq V(G)$ is a *dominating set* of G if every vertex of G is either contained in S or adjacent to a vertex of S. If instead we require that every vertex of G be adjacent to a vertex of S, then we call S a *total dominating set*. The *domination number* of G, denoted $\gamma(G)$, is the minimum cardinality of a dominating set of G, and the *total domination number* of G, $\gamma_t(G)$, is the minimum cardinality of a total dominating set of G. These two domination parameters have been extensively studied in the literature, and we refer the reader to [7,8,15].

Arumugam and Sivagnanam [2] introduced and studied the concept of neighborhood total domination in graphs. Given a graph G and a vertex v in G, the *open neighborhood* of v is the set of all vertices in G adjacent to v, and analogously, the *open neighborhood* of a set $S \subseteq V(G)$ is the set of all neighbors of vertices in S. A *neighborhood total dominating set*, abbreviated NTD-set, of G is a dominating set S of G with the property that the subgraph induced by the open neighborhood of the set S has no isolated vertex and we let $\gamma_{\mathrm{nt}}(G)$ denote the minimum cardinality of a neighborhood total dominating set of G. Recent papers on neighborhood total domination in graphs can be found in [10,11]. As every total dominating set is a NTD-set, and every NTD-set is a dominating set of G, we have the following observation first given by Arumugam and Sivagnanam in [2].

Observation 1 (*[2]*)**.** *If G is a graph with no isolated vertex, then $\gamma(G) \le \gamma_{\mathrm{nt}}(G) \le \gamma_t(G)$.*

Bounds relating the domination number of a graph G and the matching number of G, denoted $\alpha'(G)$, are studied, for example, in [3,4]. As a consequence of a result due to Bollobás and Cockayne [3], the domination number of every graph with no isolated vertex is bounded above by its matching number.

Theorem 2 (*[3]*)**.** *For every graph G with no isolated vertex, $\gamma(G) \le \alpha'(G)$.*

The total domination number versus the matching number in a graph has been studied in several papers (see, for example, [6,9,12,14,16,19,20] and elsewhere). Unlike the domination number, the total domination number and the matching number

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of a graph are generally incomparable, even for arbitrarily large, but fixed (with respect to the order of the graph), minimum degree as shown in [9]. However, there is a relationship between the total domination number, the matching number, and the path covering number of a graph G. Recall that a *path covering of* G is a collection of vertex disjoint paths such that every vertex belongs to exactly one path of G, and the cardinality of a minimum path cover is known as the *path covering number of* G, denoted $\mathrm{pc}(G)$. The following upper bound on the total domination number in terms of the matching number and path covering number is presented in [6]. This proves Graffiti.pc Conjecture #288 (see [5]).

Theorem 3 ([6]). *For every graph G with no isolated vertex, $\gamma_t(G) \le \alpha'(G) + \mathrm{pc}(G)$, and this bound is tight.*

2. Main results

We show that the bound of Theorem 3 can be improved considerably if we restrict the minimum degree, $\delta(G)$, of the graph G to be at least three.

Theorem 4. *If G is a graph with $\delta(G) \ge 3$, then $\gamma_t(G) \le \alpha'(G) + \frac{1}{2}(\mathrm{pc}(G) - 1)$, and this bound is tight.*

A proof of Theorem 4 is given in Section 3. Key to our proof is the following result on matchings and path covers in graphs.

Lemma 5. *If G is a graph of order n, then $\alpha'(G) + \frac{1}{2}\mathrm{pc}(G) \ge \frac{n}{2}$.*

The following result shows that the bound of Theorem 3 can be much improved for the neighborhood total domination number. A proof of Theorem 6 is presented in Section 5.

Theorem 6. *If G is a connected graph of order at least 3, then $\gamma_{\mathrm{nt}}(G) \le \alpha'(G) + \frac{1}{2}\mathrm{pc}(G)$ unless $G \in \{P_3, P_5, C_5\}$ in which case $\gamma_{\mathrm{nt}}(G) = \alpha'(G) + \frac{1}{2}(\mathrm{pc}(G) + 1)$.*

Embedded in the proof of the above result, we were able to classify all trees T of order n that satisfy $\alpha'(T)+\mathrm{pc}(T)/2 = n/2$. We describe the family $\mathcal{T}$ containing all such trees in Section 2.2. Furthermore, we show the following.

Theorem 7. *Let T be a tree of order $n \ge 3$. Then, $\alpha'(T) + \frac{1}{2}\mathrm{pc}(T) \ge \frac{n}{2}$ with equality if and only if $(T, S) \in \mathcal{T}$ for some labeling S.*

Theorem 8. *Let G be a connected graph of order $n \ge 3$. Then $\alpha'(G) + \frac{1}{2}\mathrm{pc}(G) = \frac{n}{2}$ if and only if G has a spanning tree T such that*
(a) *$(T, S) \in \mathcal{T}$ for some labeling S.*
(b) *$\alpha'(G) = \alpha'(T)$.*
(c) *$\mathrm{pc}(G) = \mathrm{pc}(T)$.*

The remainder of the paper is organized as follows. In Section 2.1, we give useful definitions and terminology relevant to the topics presented. We construct the family $\mathcal{T}$ described above in Section 2.2, and the proofs of Theorem 4 and Lemma 5 can be found in Section 3. Section 4 is dedicated to the proofs of Theorems 7 and 8, and Section 5 focuses on applications to other domination parameters.

2.1. Terminology and notation

For notation and graph theory terminology not defined herein, we refer the reader to [7]. Let G be a graph with vertex set $V(G)$ of order $n(G) = |V(G)|$ and edge set $E(G)$ of size $m(G) = |E(G)|$, and let v be a vertex in $V(G)$. We denote the *degree* of v in G by $d_G(v)$. The minimum degree among the vertices of G is denoted by $\delta(G)$. A vertex of degree one is called a *leaf* and its neighbor a *support vertex*. For a set $S \subseteq V(G)$, the subgraph induced by S is denoted by $G[S]$.

A *cycle* and *path* on n vertices are denoted by C_n and P_n, respectively. A *star* on $n \ge 2$ vertices is a tree with a vertex of degree $n - 1$ and is denoted by $K_{1,n-1}$. A *double star* is a tree containing exactly two vertices that are not leaves (which are necessarily adjacent). A *subdivided star* is a graph obtained from a star on at least two vertices by subdividing each edge exactly once. We note that the smallest two subdivided stars are the paths P_3 and P_5.

The *open neighborhood* of v is the set $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$ and the *closed neighborhood of* v is $N_G[v] = \{v\} \cup N_G(v)$. For a set $S \subseteq V(G)$, its *open neighborhood* is the set $N_G(S) = \bigcup_{v\in S} N_G(v)$, and its *closed neighborhood* is the set $N_G[S] = N_G(S) \cup S$. If the graph G is clear from the context, we simply write $\mathrm{d}(v)$, $N(v)$, $N[v]$, $N(S)$ and $N[S]$ rather than $d_G(v)$, $N_G(v)$, $N_G[v]$, $N_G(S)$ and $N_G[S]$, respectively. As observed in [10] a NTD-set in G is a set S of vertices such that $N[S] = V(G)$ and $G[N(S)]$ contains no isolated vertex.

A *rooted tree* T distinguishes one vertex r called the *root*. For each vertex $v \neq r$ of T, the *parent* of v is the neighbor of v on the unique (r, v)-path, while a *child* of v is any other neighbor of v. A *descendant* of v is a vertex u such that the unique (r, u)-path contains v. Let $C(v)$ and $D(v)$ denote the set of children and descendants, respectively, of v, and let $D[v] = D(v) \cup \{v\}$. A *non-leaf* of a tree T is a vertex of T of degree at least 2 in T.

Two distinct edges in a graph G are *independent* if they are not adjacent in G. A *matching* in G is a set of (pairwise) independent edges, while a matching of maximum cardinality is a *maximum matching*. The number of edges in a maximum