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Notes on a theorem of Naji

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ABSTRACT

We present a new proof of an algebraic characterization of circle graphs due to W. Naji. For bipartite graphs, Naji's theorem is equivalent to an algebraic characterization of planar matroids due to J. Geelen and B. Gerards. Naji's theorem also yields an algebraic characterization of permutation graphs.

1. Introduction

This paper is concerned with the following notion.

Definition 1. Let $W = w_1 \dots w_{2n}$ be a double occurrence word in the letters $v_1, \dots, v_n$. The *interlacement graph* $\mathcal{I}(W)$ is the simple graph with vertex-set $V = \{v_1, \dots, v_n\}$, in which v_i and v_j are adjacent if and only if they are *interlaced* in W, i.e., they appear in W in the order $v_iv_jv_iv_j$ or $v_jv_iv_jv_i$. A *circle graph* is a simple graph that can be realized as the interlacement graph of some double occurrence word.

As far as we know, the idea of interlacement first appeared in the form of a symmetric matrix used in Brahana's 1921 study of curves on surfaces [8]. Interlacement graphs were studied by Zelinka [27], who credited the idea to Kotzig. During the subsequent decades several researchers discussed graphs and matrices defined using interlacement. Cohn and Lempel [9] and Even and Itai [16] used them to analyze permutations, and Bouchet [3] and Read and Rosenstiehl [25] used them to study Gauss' problem of characterizing generic self-intersecting curves in the plane. Recognition algorithms for circle graphs have been introduced by Bouchet [4], Gioan, Paul, Tedder and Corneil [21], Naji [23,24] and Spinrad [26].

Although Naji's is not the best of the circle graph recognition algorithms in terms of computational complexity, it is particularly interesting for two reasons. The first reason is that Naji's characterization is only indirectly algorithmic; it involves a system of equations that may be defined for any graph, which is only solvable for circle graphs. The second reason is that the two known proofs of the theorem are quite long. The original argument ends on p. 173 of Naji's thesis [23]. A much shorter argument was given by Gasse [18], but Gasse's argument requires Bouchet's circle graph obstructions theorem [6], which itself has a long and difficult proof.

A couple of years ago, Geelen and Gerards [20] characterized graphic matroids by a system of equations that resembles Naji's system of equations. (Indeed, they mention that Naji's theorem motivated their result.) The resemblance is limited to the equations; there is a striking contrast between their concise, well-motivated proof and Naji's long, detailed argument. This contrast encouraged us to look for an alternative proof of Naji's theorem; we eventually developed the one presented

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below. Although our argument is certainly not as elegant as the proof of Geelen and Gerards, it is shorter than either Naji's original proof or the combination of a proof of Bouchet's obstructions theorem and Gasse's derivation of Naji's theorem.

In addition to proving Naji's theorem for circle graphs in general, at the end of the paper we briefly discuss two special cases. First, the restriction of Naji's theorem to bipartite graphs is equivalent to the restriction of the Geelen–Gerards characterization to planar matroids. Second, Naji's theorem also characterizes permutation graphs.

Before proceeding we should thank Jim Geelen for his comments on Naji's theorem. In particular, he pointed out that although all circle graphs have solutions of Naji's equations that arise naturally from double occurrence words, some circle graphs also have other Naji solutions that do not seem so natural. He conjectured that these other solutions might correspond in some way to splits. (See Sections 2 and 3 for definitions, and Section 5 for examples.) Although we do not address Geelen's conjecture directly we do provide some indirect evidence for it, as the first step of our proof of Naji's theorem involves showing that none of these other solutions occur in circle graphs that have no splits. (See Section 6.) We should also thank an anonymous reader, whose comments led to Corollary 26 and several other improvements in the paper.

2. Naji's equations and their solutions

We begin with some definitions.

Definition 2 (*[23,24]*)**.** Let G be a simple graph. For each pair of distinct vertices v and w of G, let $\beta(v,w)$ and $\beta(w,v)$ be distinct variables. Then the *Naji equations* for G are the following.

(a) For each edge vw of G, $\beta(v,w)+\beta(w,v)=1$.

(b) If v,w,x are three distinct vertices of G such that $vw \in E(G)$ and $vx, wx \notin E(G)$, then $\beta(x,v)+\beta(x,w)=0$.

(c) If v,w,x are three distinct vertices of G such that $vw, vx \in E(G)$ and $wx \notin E(G)$, then $\beta(v,w)+\beta(v,x)+\beta(w,x)+\beta(x,w)=1$.

If the Naji equations of G have a solution over $GF(2)$, the field with two elements, then any such solution is a *Naji solution* and G is a *Naji graph*. We use the following notation:

Definition 3. If G is a graph then $\mathcal{B}(G)$ denotes the set of Naji solutions of G.

Of course G is a Naji graph if and only if $\mathcal{B}(G) \neq \varnothing$, and elementary linear algebra guarantees that if $\mathcal{B}(G) \neq \varnothing$ then $|\mathcal{B}(G)| = 2^k$ for some $k \geq 0$. In particular, if $n=1$ then G is a Naji graph and $\mathcal{B}(G)=\{\varnothing\}$.

Notice that the three types of Naji equations are distinct. An equation of type (a) involves only two vertices, an equation of type (b) involves no nonzero constant and an equation of type (c) has four terms. For this reason, when discussing the Naji equations we do not always cite a specific type of equation. We might also mention two obvious consequences of the equations, which will be useful. First: the type (b) equations imply that $\beta(x,-)$ is constant on each connected component of $G-N(x)$. (Here $N(x)$ denotes the open neighborhood of x, $N(x)=\{y \in V(G) \mid xy \in E(G)\}$.) Second: variants of a type (c) equation are obtained by replacing $\beta(v,w)$ or $\beta(v,x)$ with $\beta(w,v)+1$ or $\beta(x,v)+1$ (respectively), and then moving each new $+1$ to the right hand side.

Naji's theorem [23,24] states that G is a Naji graph if and only if G is a circle graph. One direction of Naji's theorem is easy.

Proposition 4. *Every circle graph is a Naji graph.*

Proof. Consider a double occurrence word W. An *orientation* of W is given by arbitrarily designating one appearance of each letter as "initial"; the other appearance is "terminal". We use the notation v^{in} and v^{out} for the initial and terminal appearances of v, respectively. For each orientation of W, define a function β by: $\beta(v,w)=0$ if and only if when we cyclically permute W to begin with v^{in}, w^{out} precedes v^{out}.

We claim that this β is a Naji solution of G. If $vw \in E(G)$ then after cyclically permuting W to begin with v^{in}, W will be in the form $v^{in}\ldots w^{in}\ldots v^{out}\ldots w^{out}\ldots$ or in the form $v^{in}\ldots w^{out}\ldots v^{out}\ldots w^{in}\ldots$. In the first case, $\beta(v,w)+\beta(w,v)=1+0$ and in the second case, $\beta(v,w)+\beta(w,v)=0+1$. For the type (b) equations, if $vw \in E(G)$ and $vx, wx \notin E(G)$ then after cyclically permuting W to begin with x^{in}, and interchanging v and w (if necessary) so that v appears before w, W will be in one of these forms.

$$x^{in}\ldots v\ldots w\ldots v\ldots w\ldots x^{out}\ldots \qquad x^{in}\ldots x^{out}\ldots v\ldots w\ldots v\ldots w\ldots.$$

In the first case $\beta(x,v)+\beta(x,w)=1+1$, and in the second case $\beta(x,v)+\beta(x,w)=0+0$. For the type (c) equations, if $vw, vx \in E(G)$ and $wx \notin E(G)$ then after cyclically permuting W to begin with v^{in}, and interchanging w and x (if necessary) so that w appears before x, we may presume W is in one of these forms:

$$\begin{array}{ll} v^{in}\ldots w^{in}\ldots x^{in}\ldots v^{out}\ldots x^{out}\ldots w^{out}\ldots & v^{in}\ldots w^{out}\ldots x^{in}\ldots v^{out}\ldots x^{out}\ldots w^{in}\ldots \\ v^{in}\ldots w^{in}\ldots x^{out}\ldots v^{out}\ldots x^{in}\ldots w^{out}\ldots & v^{in}\ldots w^{out}\ldots x^{out}\ldots v^{out}\ldots x^{in}\ldots w^{in}\ldots. \end{array}$$

Proceeding from left to right, the sum $\beta(v,w)+\beta(v,x)+\beta(w,x)+\beta(x,w)$ is $1+1+0+1$ or $0+1+1+1$ for the words in the top row, and $1+0+0+0$ or $0+0+1+0$ for the words in the bottom row. ■