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# On perfection and imperfection of one-realizations of a given set ☆

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ABSTRACT

For any set $S$ of positive integers, a mixed hypergraph $\mathcal{H}$ is a one-realization of $S$ if its feasible set is $S$ and each entry of its chromatic spectrum is either 0 or 1. In this paper, we focus on several special kinds of one-realizations of a given set $S$ and get the following results: each colorable mixed hypergraph can be embedded into a one-realization of $S$; there exist one-realizations of $S$ which are $\mathcal{C}$-perfect; furthermore, for any positive integer $k$, there exists a one-realization $\mathcal{H}$ of $S$ with $\alpha_{\mathcal{C}}(\mathcal{H}) - \overline{\chi}(\mathcal{H}) > k$.


## 1. Introduction

A *mixed hypergraph* on a set $X$ is a triple $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$, where $\mathcal{C}$ and $\mathcal{D}$ are families of subsets of $X$, called the *$\mathcal{C}$-edges* and *$\mathcal{D}$-edges*, respectively. The members in $\mathcal{C} \cap \mathcal{D}$ are called *bi-edges*, and $\mathcal{H}$ is called a *bi-hypergraph* if $\mathcal{C} = \mathcal{D} = \mathcal{B}$, in which case we may simply write $\mathcal{H} = (X, \mathcal{B})$. A sub-hypergraph $\mathcal{H}' = (X', \mathcal{C}', \mathcal{D}')$ of $\mathcal{H}$ is a *spanning sub-hypergraph* if $X' = X$, and $\mathcal{H}'$ is called an *induced sub-hypergraph* of $\mathcal{H}$ on $X'$, denoted by $\mathcal{H}[X']$, when $\mathcal{C}' = \{C \in \mathcal{C} \mid C \subseteq X'\}$ and $\mathcal{D}' = \{D \in \mathcal{D} \mid D \subseteq X'\}$. On the other hand, if a mixed hypergraph $\mathcal{H}'$ is an induced sub-hypergraph of $\mathcal{H}$, we also say that $\mathcal{H}'$ can be *embedded into* $\mathcal{H}$. Two mixed hypergraphs $\mathcal{H}_1 = (X_1, \mathcal{C}_1, \mathcal{D}_1)$ and $\mathcal{H}_2 = (X_2, \mathcal{C}_2, \mathcal{D}_2)$ are *isomorphic* if there exists a bijection $\phi$ between $X_1$ and $X_2$ that maps each $\mathcal{C}$-edge of $\mathcal{C}_1$ onto a $\mathcal{C}$-edge of $\mathcal{C}_2$ and maps each $\mathcal{D}$-edge of $\mathcal{D}_1$ onto a $\mathcal{D}$-edge of $\mathcal{D}_2$, and vice versa. The bijection $\phi$ is called an *isomorphism* from $\mathcal{H}_1$ to $\mathcal{H}_2$. In general, we follow the terminologies of [22].

The difference between $\mathcal{C}$-edges and $\mathcal{D}$-edges lies in the requirement on colorings. A *proper $k$-coloring* of $\mathcal{H}$ is a mapping from $X$ into a set of $k$ *colors* so that each $\mathcal{C}$-edge has two vertices with a *Common* color and each $\mathcal{D}$-edge has two vertices with *Distinct* colors. Different from the traditional coloring of graphs and hypergraphs, a mixed hypergraph may admit no colorings at all. We call $\mathcal{H}$ *colorable* if it admits at least one proper coloring; otherwise it is *uncolorable*. Assume that $\mathcal{H}$ is colorable. By a *strict $k$-coloring* we mean a proper vertex coloring with exactly $k$ colors; that is, a coloring $\varphi : X \to \mathbb{N}$ with

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$|\varphi(X)| = k$. The set

$$\Phi(\mathcal{H}) := \{k \mid \mathcal{H} \text{ has a strict } k\text{-coloring}\}$$

is termed the *feasible set* of $\mathcal{H}$. The *lower chromatic number* $\chi(\mathcal{H})$ of $\mathcal{H}$ is the smallest number contained in $\Phi(\mathcal{H})$, and the *upper chromatic number* $\overline{\chi}(\mathcal{H})$ of $\mathcal{H}$ is the largest number contained in $\Phi(\mathcal{H})$. Each strict $k$-coloring $\varphi$ of $\mathcal{H}$ induces a *feasible partition* $X_1 \cup X_2 \cup \cdots \cup X_k = X$, where the partition classes are the monochromatic subsets of $X$ under $\varphi$. For every $k$, let $r_k$ denote the number of feasible partitions of $X$ into $k$ nonempty classes. The vector $R(\mathcal{H}) = (r_1, r_2, \ldots, r_{\overline{\chi}})$ is called the *chromatic spectrum* of $\mathcal{H}$.

The study of the colorings of mixed hypergraphs has made a lot of progress since its inception [23]. The applications of mixed hypergraph coloring include modeling of several types of graph coloring (like list coloring without lists, that is to say, list colorings of graphs can be modeled as colorings of mixed hypergraphs without any lists [21]), different kinds of homomorphism of simple graphs and multigraphs, channel assignment problems [17] and the newest application in problems arising in cyber security [15]. Bujtás and Tuza [2–5,9,6] gave a generalization and introduced the concept of color-bounded hypergraph; and Dvořák et al. [14] unified several previously studied colorings and introduced the concept of pattern hypergraphs. Recently, in Chapter 11: "Hypergraph Colouring" of [1], Cs. Bujtás, Zs. Tuza and V. Voloshin briefly surveyed the half-century history of hypergraph coloring. For more information, see [22] and the regularly updated website [24].

One of the most surprising properties of a mixed hypergraph is that its chromatic spectrum may have gaps. A *gap* in the chromatic spectrum of $\mathcal{H}$, or a gap in $\Phi(\mathcal{H})$, is an integer $k \notin \Phi(\mathcal{H})$ such that $\min \Phi(\mathcal{H}) < k < \max \Phi(\mathcal{H})$. If $\Phi(\mathcal{H})$ has no gaps, then the chromatic spectrum is termed *continuous* or *gap-free*; otherwise it is said to be *broken*. More generally, a gap of *size l* in $\Phi(\mathcal{H})$ means $l$ consecutive integers, each larger than $\chi(\mathcal{H})$ and smaller than $\overline{\chi}(\mathcal{H})$, that are all missing in $\Phi(\mathcal{H})$.

For a finite set $S$ of positive integers, we say that a mixed hypergraph $\mathcal{H}$ is a *realization* of $S$ if $\Phi(\mathcal{H}) = S$, and $\mathcal{H}$ is a *one-realization* of $S$ if it is a realization of $S$ and each entry of its chromatic spectrum is either 0 or 1. It is readily seen that if $1 \in \Phi(\mathcal{H})$, then $\mathcal{H}$ cannot have any $\mathcal{D}$-edges. For the case $1 \notin \Phi(\mathcal{H})$, Jiang et al. [16] proved that for any finite set $S$ of integers greater than 1, there exists a mixed hypergraph $\mathcal{H}$ such that $\Phi(\mathcal{H}) = S$. Král [18] strengthened this result by showing that prescribing any sequence of positive integers $r_k$, $k \in S$, there exists a mixed hypergraph which has precisely $r_k$ strict $k$-colorings for all $k \in S$. Bujtás and Tuza [7] focused on the feasible set of uniform mixed hypergraphs and proved that for every integer $r \geq 3$, a set of positive integers is the feasible set of an $r$-uniform mixed hypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ with $|\mathcal{C}| + |\mathcal{D}| \geq 1$ if and only if (i) $\min S \geq r$, or (ii) $2 \leq \min S \leq r - 1$ and $S$ contains all integers between $\min S$ and $r - 1$, or (iii) $\min S = 1$ and $S = \{1, \ldots, \overline{\chi}\}$ for some natural number $\overline{\chi} \geq 1$. Zhao et al. [27] strengthened this result by showing that prescribing any sequence of positive integers $r_k$, $k \in S$, there exists a 3-uniform bi-hypergraph which has precisely $r_k$ strict $k$-colorings for all $k \in S$. Enormous number of papers are focused on the minimum number of vertices or edges of realizations or one-realizations of $S$, e.g., [11–13,18,20,16,25–28].

In graph coloring theory, perfect graphs provide an important theoretic and algorithmic topic of research. A graph $G$ is called *perfect* if, for every one of its induced subgraphs $G'$ (including $G$ itself), the lower chromatic number equals the size of its largest clique, i.e., $\chi(G') = \omega(G')$. The notion of graph perfection is difficult to extend to general $\mathcal{D}$-hypergraphs because, from the point of view of colorings, there is no natural analogue of complete graphs. In contrast, there is a natural notion of perfection of an arbitrary mixed hypergraph with respect to the upper chromatic number. Let $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ be a mixed hypergraph. A set $Y \subseteq X$ is said to be $\mathcal{C}$*-stable* if it contains no $\mathcal{C}$-edge, and the $\mathcal{C}$*-stability number* $\alpha_{\mathcal{C}}(\mathcal{H})$ is the cardinality of a maximum $\mathcal{C}$-stable set. Obviously, $\overline{\chi}(\mathcal{H}) \leq \alpha_{\mathcal{C}}(\mathcal{H})$. A colorable mixed hypergraph is called $\mathcal{C}$*-perfect* if $\overline{\chi}(\mathcal{H}') = \alpha_{\mathcal{C}}(\mathcal{H}')$ holds for every induced sub-hypergraph $\mathcal{H}'$ (including $\mathcal{H}$ itself). The perfection of mixed hypergraphs has attracted a lot of attention, e.g., [10,8,19,23].

In this paper, we focus on some special kinds of one-realizations of $S$ and get the following results: each colorable mixed hypergraph can be embedded into a one-realization of $S$; there exist one-realizations of $S$ which are $\mathcal{C}$-perfect; furthermore, there exist one-realizations with arbitrarily large difference between the $\mathcal{C}$-stability number and the upper chromatic number.

In the following, we always assume that $s \geq 2$ is an integer and $S := \{n_1, n_2, \ldots, n_s\}$ is a finite set of integers with $n_1 > n_2 > \cdots > n_s \geq 2$. For any positive integer $n$, let $[n]$ denote the set $\{1, 2, \ldots, n\}$.

## 2. Preliminaries

In this section, we first introduce a family of one-realizations of $S$ which is important in getting our main results, then compare the colorings of a mixed hypergraph and its sub-hypergraphs.

**Basic Construction.** (Zhao et al. [28]) For each $k \in [s] \setminus \{1\}$, let $M_k = \{0, 1, \ldots, n_{k-1} - n_k - 1\}$ and set

$$X_{n_1,\ldots,n_s} = \{\theta_i \mid i \in [n_s]\} \cup \{\eta_{n_k+p}, \gamma_{n_k+p} \mid k \in [s] \setminus \{1\}, p \in M_k\} \cup \{\eta_{n_1}\}, \quad \text{where}$$

$$\theta_i = (\underbrace{i, \ldots, i}_{s}), \qquad \eta_{n_1} = (n_1, n_2, \ldots, n_s) \quad \text{and}$$

$$\eta_{n_k+p} = (\underbrace{n_k + p, \ldots, n_k + p}_{k-1}, n_k, n_{k+1}, \ldots, n_s),$$