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Abacus-tournament models for Hall–Littlewood polynomials

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ABSTRACT

In 2010, the first author introduced a combinatorial model for Schur polynomials based on labeled abaci. We generalize this construction to give analogous models for the Hall–Littlewood symmetric polynomials P_λ, Q_λ, and R_λ using objects called abacus-tournaments. We introduce various cancellation mechanisms on abacus-tournaments to obtain simpler combinatorial formulas and explain why these polynomials are divisible by certain products of t-factorials. These tools are then applied to give bijective proofs of several identities involving Hall–Littlewood polynomials, including the Pieri rule that expands the product $P_\mu e_k$ into a linear combination of Hall–Littlewood polynomials.

1. Introduction

The *Hall–Littlewood polynomials* $P_\lambda(x_1, \ldots, x_N; t)$ are symmetric polynomials in the variables $x_1, \ldots, x_N$ that involve an extra parameter t. These polynomials arose indirectly in Philip Hall's study [5] of the subgroup lattices of finite abelian p-groups; Littlewood later gave an explicit definition [7]. The Hall–Littlewood polynomials specialize to the *Schur symmetric polynomials* s_λ when $t = 0$, and they specialize to the *monomial symmetric polynomials* m_λ when $t = 1$. The connection to group theory arises as follows. Each product $P_\mu P_\nu$ has a unique expansion $\sum_\lambda f^\lambda_{\mu,\nu}(t)P_\lambda$, where the structure constants $f^\lambda_{\mu,\nu}(t)$ are polynomials in t with integer coefficients. If p is prime and M is a finite abelian p-group of type λ, then $f^\lambda_{\mu,\nu}(p^{-1})$ multiplied by an appropriate power of p counts the number of subgroups N of M such that N has type μ and M/N has type ν. Macdonald's monograph [11, Ch. I—III] provides a thorough treatment of these topics from the algebraic viewpoint.

In the last decade or so, many researchers have worked to uncover the combinatorial significance of various symmetric polynomials. A rich theory of tableau combinatorics has been developed to provide bijective explanations of many identities involving Schur symmetric polynomials; see, for instance, the texts of Fulton [4], Sagan [12], or Stanley [13]. The vector space of homogeneous symmetric polynomials of a fixed degree has many interesting bases, and the transition matrices between these bases often encode important combinatorial information. This viewpoint is explored, for instance, in [1,3]. Some transition matrices involving Hall–Littlewood polynomials are studied in [10] using the combinatorics of starred semistandard tableaux. Carbonara [2] developed a model for the transition matrix expanding P_λ in terms of the Schur polynomials involving objects called *special tournament matrices*.

More recently, the first author [8] formulated a combinatorial model for antisymmetrized Schur polynomials based on *labeled abaci*, which were inspired by the unlabeled abaci of James and Kerber [6]. This abacus model was used to give

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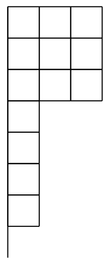

Fig. 1. Ferrers diagram of a partition.

short bijective proofs of results such as the Pieri rules for Schur polynomials, the Littlewood–Richardson rule, and the equivalence of the algebraic definition and the combinatorial definition of Schur polynomials. Our goal in this paper is to undertake an analogous study of antisymmetrized Hall–Littlewood polynomials using objects that combine labeled abaci and tournaments.

We study three versions of Hall–Littlewood symmetric polynomials, denoted P_λ, Q_λ, and R_λ, which can be turned into antisymmetric polynomials by multiplying by the Vandermonde determinant $a_{\delta(N)}$ (see Section 2 for precise definitions). We develop several combinatorial models for these antisymmetrized Hall–Littlewood polynomials as sums of signed, weighted collections of *abacus-tournaments*. In particular, the concepts of *global* and *local* exponent collisions enable us to streamline our initial model for $a_{\delta(N)}R_\lambda$ in various ways by canceling pairs of objects with the same weight and opposite sign. To pass from R_λ to P_λ or Q_λ, we must also divide the polynomial by certain products of t-factorials. This division is explained combinatorially by the notion of abacus-tournaments that are *leading* in certain positions, which is closely related to Carbonara's restriction to "special" tournaments in his work on the inverse t-Kostka matrix [2].

The second half of the paper uses our combinatorial models to give bijective proofs of some identities involving Hall–Littlewood polynomials. The deepest result is an abacus-tournament proof of one of the Pieri rules, which tells how the product of P_μ with an elementary symmetric function e_k can be expanded as a linear combination of P_λ's. We will see that delicate interactions between various cancellation mechanisms provide an elegant bijective explanation of this algebraic identity.

The paper is organized as follows. Section 2 recalls the definitions of Schur polynomials, Hall–Littlewood polynomials, and ancillary combinatorial constructs. Section 3 defines abacus-tournaments and develops several models for $a_{\delta(N)}R_\lambda$. Section 4 builds up more combinatorial tools leading to formulas for $a_{\delta(N)}P_\lambda$ and $a_{\delta(N)}Q_\lambda$. Section 5 illustrates how these ideas can be applied to prove some basic identities involving Hall–Littlewood polynomials. Section 6 gives an abacus-tournament proof of the Pieri rule for multiplying a Hall–Littlewood polynomial by e_k. Section 7 indicates some planned future directions for this line of research. We remark that all results in the current paper are based upon the second author's doctoral dissertation [14].

2. Background

This section reviews the background material on partitions, permutations, and symmetric polynomials needed to define Hall–Littlewood polynomials. More systematic expositions of this material may be found (for instance) in [9,11,13].

2.1. Partitions

A *partition* is a weakly decreasing sequence of nonnegative integers. For each positive integer N, let Par_N be the set of partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N)$ with exactly N parts, some of which may be zero. Given $\lambda \in \text{Par}_N$, let $|\lambda| = \lambda_1+\lambda_2+\cdots+\lambda_N$, and let $\ell(\lambda)$ be the number of *nonzero* parts of λ. For $d \geq 0$, let $\text{Par}_N(d) = \{\lambda \in \text{Par}_N : |\lambda| = d\}$. Finally, given $\lambda \in \text{Par}_N$ and an integer $j \geq 0$, let $m_j(\lambda)$ be the number of parts of λ equal to j. For example, $\lambda = (3, 3, 3, 1, 1, 1, 1, 0)$ is a partition in $\text{Par}_8(13)$ with $\ell(\lambda) = 7$, $m_0(\lambda) = 1$, $m_1(\lambda) = 4$, $m_3(\lambda) = 3$, and $m_j(\lambda) = 0$ for all other j.

We can visually represent a partition λ by its *Ferrers diagram*, which consists of N left-justified rows of boxes with λ_i boxes in the ith row from the top. Our example partition has the diagram shown in Fig. 1. The vertical line in the last row of the diagram corresponds to the part of size zero in λ; we include this line so that N (the number of parts) can be read off from the figure.

For any partition $\lambda \in \text{Par}_N$, we let λ'_j be the number of boxes in the jth column of the Ferrers diagram of λ. Also define $\lambda'_0 = N$; then $m_j(\lambda) = \lambda'_j - \lambda'_{j+1}$ for all $j \geq 0$. Our example partition has $\lambda'_0 = 8$, $\lambda'_1 = 7$, $\lambda'_2 = \lambda'_3 = 3$, and $\lambda'_j = 0$ for all $j \geq 4$.

2.2. Permutations, t-factorials, and t-binomial coefficients

A *permutation* of the set $[N] = \{1, 2, \ldots, N\}$ is a bijection $w : [N] \to [N]$. The *symmetric group* S_N is the set of all permutations of $[N]$ under the operation of function composition. We identify a permutation $w \in S_N$ with the word