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Note

Adjacent vertex distinguishing total colorings of 2-degenerate graphs

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ABSTRACT

Let ϕ be a proper total coloring of G. We use $C_\phi(v) = \{\phi(v)\} \cup \{\phi(uv) \mid uv \in E(G)\}$ to denote the set of colors assigned to a vertex v and those edges incident with v. An adjacent vertex distinguishing total coloring of a graph G is a proper total coloring of G such that $C_\phi(u) \neq C_\phi(v)$ for any $uv \in E(G)$. The minimum number of colors required for an adjacent vertex distinguishing total coloring of G is denoted by $\chi''_a(G)$. In this paper we show that if G is a 2-degenerate graph, then $\chi''_a(G) \leq \max\{\Delta(G) + 2, 6\}$. Moreover, we also show that when $\Delta \geq 5$, $\chi''_a(G) = \Delta(G) + 2$ if and only if G contains two adjacent vertices of maximum degree. Our results imply the results on outerplanar graphs (Wang and Wang, 2010), K_4-minor free graphs (Wang and Wang, 2009) and graphs with maximum average degree less than 3 (Wang and Wang, 2008).

1. Introduction

In this paper we only consider simple graphs. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. We use $d_G(v)$ to denote the degree of a vertex v in G. A *k-vertex*, *k^--vertex*, *k^+-vertex* is a vertex of degree k, at most k, at least k, respectively. We call k-vertices, k^+-vertices adjacent to v *k-neighbors*, *k^+-neighbors* of v, respectively. Let $\Delta(G)$ be the maximum degree of G. If there is no confusion from the context we use simply Δ. To identify two vertices u and v of a graph G is to replace these vertices by a single vertex incident to all the edges which were incident in G to either u or v.

A *proper total k-coloring* is a mapping $\phi : V(G) \cup E(G) \longrightarrow \{1, 2, \ldots, k\}$ such that any two adjacent or incident elements in $V(G) \cup E(G)$ receive different colors. The *total chromatic number* $\chi''(G)$ of G is the smallest integer k such that G has a total k-coloring. Let ϕ be a total coloring of G. We use $C_\phi(v) = \{\phi(v)\} \cup \{\phi(uv) \mid uv \in E(G)\}$ to denote the set of colors assigned to a vertex v and those edges incident with v. A proper total k-coloring ϕ of G is *adjacent vertex distinguishing*, or a *total-k-avd-coloring*, if $C_\phi(u) \neq C_\phi(v)$ whenever $uv \in E(G)$. The *adjacent vertex distinguishing total chromatic number* $\chi''_a(G)$ is the smallest integer k such that G has a total-k-avd-coloring. It is obvious that $\Delta + 1 \leq \chi''(G) \leq \chi''_a(G)$. Note that if a graph G contains two adjacent vertices of maximum degree, then $\chi''_a(G) \geq \Delta + 2$.

This coloring related to vertex-distinguishing proper edge colorings of graphs was first examined by Burris and Schelp [3] and was further discussed by many others, including Bazgan et al. [2] and Balister et al. [1]. This type of coloring was later extended to require only adjacent vertices to be distinguished by Zhang et al. [14], which was in turn extended to proper total colorings [13].

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Zhang et al. [13] determined $\chi_a''(G)$ for some basic graphs such as paths, cycles, fans, wheels, trees, complete graphs, and complete bipartite graphs and made the following conjecture in terms of the maximum degree $\Delta(G)$.

Conjecture 1.1. *For any graph G, $\chi_a''(G) \le \Delta(G) + 3$.*

Chen [4] and Wang [7], independently, confirmed Conjecture 1.1 for graphs G with $\Delta \le 3$. Later, Hulgan [6] presented a more concise proof on this result. Wang and Huang [9] proved Conjecture 1.1 for planar graphs with $\Delta \ge 11$. Huang et al. [5] proved that $\chi_a''(G) \le 2\Delta$ in general.

A graph G is called K_4-minor free if G does not have K_4 as a minor. A planar graph is said to be outerplanar if it has a plane embedding such that all vertices lie on the boundary of the unbounded face. Wang et al. considered the adjacent vertex distinguishing total chromatic number of K_4-minor free graphs [11] and outerplanar graphs [12].

Theorem 1.2. *Let G be a K_4-minor free graph with $\Delta \ge 3$. Then $\Delta + 1 \le \chi_a''(G) \le \Delta + 2$ and $\chi_a''(G) = \Delta + 2$ if and only if G contains two adjacent Δ-vertices.*

Theorem 1.3. *Let G be an outerplanar graph with $\Delta \ge 3$. Then $\Delta + 1 \le \chi_a''(G) \le \Delta + 2$ and $\chi_a''(G) = \Delta + 2$ if and only if G contains two adjacent Δ-vertices.*

The *average degree* of a graph G is $\frac{2|E(G)|}{|V(G)|}$. The *maximum average degree*, $mad(G)$, of G is the maximum of the average degrees of its subgraphs. In [10], Wang proved the following theorem.

Theorem 1.4. *Let G be a graph with $mad(G) < 3$.*

(i) *If $\Delta \ge 5$, then $\Delta + 1 \le \chi_a''(G) \le \Delta + 2$ and $\chi_a''(G) = \Delta + 2$ if and only if G contains two adjacent Δ-vertices.*
(ii) *If $\Delta \le 4$, then $\chi_a''(G) \le 6$.*

A graph G is called 2-*degenerate* if every subgraph of G contains a vertex of degree at most 2. Note that outerplanar graphs, K_4-minor free graphs and graphs with maximum average degree less than 3 are all 2-degenerate graphs. In [8], Wang et al. considered the adjacent vertex distinguishing edge colorings of 2-degenerate graphs.

In this paper, by taking a complete different approach, we prove the following result for 2-degenerate graphs which implies Theorem 1.4, and implies Theorems 1.2 and 1.3 partly. We also characterize the 2-degenerate graphs having $\chi_a''(G) = \Delta + 2$ for $\Delta \ge 5$.

Theorem 1.5. *Let G be a 2-degenerate graph. Then*

(i) *$\chi_a''(G) \le 6$ if $\Delta \le 4$.*
(ii) *$\chi_a''(G) \le \Delta + 2$ for $\Delta \ge 5$, and $\chi_a''(G) = \Delta + 2$ if and only if G contains two adjacent Δ-vertices.*

For a graph G, let $k(G) = \max\{\Delta + 2, 6\}$ if G contains two adjacent Δ-vertices and $k(G) = \max\{\Delta + 1, 6\}$ otherwise. Then $k(G) \ge 6$. Thus Theorem 1.5 is equivalent to the following theorem.

Theorem 1.6. *Let G be a 2-degenerate graph. Then $\chi_a''(G) \le k(G)$.*

2. Proof of Theorem 1.6

Lemma 2.1 ([13]). *Let P_n be a path of order $n \ge 2$ and C_n be a cycle of order $n \ge 3$. Then*

(i) *$\chi_a''(P_n) = 3$ if $2 \le n \le 3$, and $\chi_a''(P_n) = 4$ otherwise.*
(ii) *$\chi_a''(C_n) = 5$ if $n = 3$, and $\chi_a''(C_n) = 4$ otherwise.*

Proof of Theorem 1.6. Suppose to the contrary that G is a counterexample to Theorem 1.6 such that $|E(G)|$ is minimum. Then G is connected. If $\Delta = 1$, then $G = P_2$. If $\Delta = 2$, then G is a path or a cycle. By Lemma 2.1, we may assume that $\Delta \ge 3$. Denote $k = k(G)$ and $[k] = \{1, 2, \ldots, k\}$ the set of colors.

Claim 2.1. *For every subgraph H of G with $|E(H)| < |E(G)|$, H has a total-k-avd-coloring.*

Proof of Claim 2.1. By the choice of G, for any subgraph H of G with $|E(H)| < |E(G)|$, H has a total-$k(H)$-avd-coloring. Since $k(H) \le k(G) = k$, then H has a total-k-avd-coloring.

Claim 2.2. *No 2-vertex is adjacent to a 2^--vertex.*