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Note

Existence of rainbow matchings in strongly edge-colored graphs

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ABSTRACT

The famous Ryser Conjecture states that there is a transversal of size n in a latin square of odd order n, which is equivalent to finding a rainbow matching of size n in a properly edge-colored $K_{n,n}$ when n is odd. Let δ denote the minimum degree of a graph. In 2011, Wang proposed a more general question to find a function $f(\delta)$ ($f(\delta) \geq 2\delta + 1$) such that for each properly edge-colored graph of order $f(\delta)$, there exists a rainbow matching of size δ. The best bound so far is $f(\delta) \leq 3.5\delta + 2$ due to Lo. Babu et al. considered this problem in strongly edge-colored graphs in which each path of length 3 is rainbow. They proved that if G is a strongly edge-colored graph of order at least $2\lfloor \frac{3\delta}{4} \rfloor$, then G has a rainbow matching of size $\lfloor \frac{3\delta}{4} \rfloor$. They proposed an interesting question: Is there a constant c greater than $\frac{3}{4}$ such that every strongly edge-colored graph G has a rainbow matching of size at least $c\delta$ if $|V(G)| \geq 2\lfloor c\delta \rfloor$? Clearly, $c \leq 1$. We prove that if G is a strongly edge-colored graph with minimum degree δ and order at least $2\delta + 2$, then G has a rainbow matching of size δ.

1. Introduction

We use [5] for terminology and notations not defined here and consider simple undirected graphs only. Let $G = (V, E)$ be a graph. For a subgraph H of G, let $|H|$ denote the *order* of H, i.e. the number of vertices of H and let $\|H\|$ denote the *size* of H, i.e. the number of edges of H. Let δ denote the *minimum degree* of a graph G and $n = |G|$.

A subgraph in an edge-colored graph is called *rainbow* if all its edges have distinct colors. Recently rainbow matchings in graphs and hypergraphs have been received much attention, see [1,2,4,11,12]. The study of rainbow matchings originates from the famous Ryser Conjecture [9], which states that there is a transversal of size n in a latin square of odd order n. Note that this problem is equivalent to finding a rainbow matching of size n in a properly edge-colored $K_{n,n}$ when n is odd. In [17], Wang proposed a more general question: Is there a function $f(\delta)$ such that every properly edge-colored graph of order $f(\delta)$ contains a rainbow matching of size δ? Diemunsch et al. [6] showed that such function does exist and $f(\delta) \leq 98\delta/23$. Gyárfás and Sárközy [8] improved the result to $f(\delta) \leq 4\delta - 3$. Independently, Tan and Lo [15] showed that $f(\delta) \leq 4\delta - 4$ for $\delta \geq 4$. Now the best result is $f(\delta) \leq 3.5\delta + 2$ due to Lo [14]. In fact, he proved this result in the more general setting of color degree conditions, which have also been extensively studied, see [10,13,18].

Since lower bounds for the size of maximum rainbow matchings in properly edge-colored graphs have attracted much attention, it is natural to try to improve the lower bounds under stronger assumptions on the edge-coloring. A properly edge-colored graph is a graph such that every path of length 2 is rainbow. A strongly edge-colored graph is a graph such that

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every path of length 3 is rainbow. The study of strong edge colorings of graphs is an active topic in coloring theory [7,16]. Rainbow matchings in strongly edge-colored graphs have an interpretation that seems to be intuitively closer to that of rainbow matchings in properly edge-colored graphs, than with the other strengthenings of proper coloring like acyclic edge-coloring and star edge-coloring. In [3], Babu et al. showed that if G is a strongly edge-colored graph of order at least $2\lfloor\frac{3\delta}{4}\rfloor$, then G has a rainbow matching of size $\lfloor\frac{3\delta}{4}\rfloor$, otherwise $\lfloor\frac{|V(G)|}{2}\rfloor$. They proposed an interesting question: Is there a constant c greater than $\frac{3}{4}$ such that every strongly edge-colored graph G has a rainbow matching of size at least $c\delta$ if $|V(G)| \geq 2\lfloor c\delta\rfloor$? Clearly, $c \leq 1$. In this paper, we almost answer this question and prove the following result.

Theorem 1.1. *If G is a strongly edge-colored graph with minimum degree δ and order at least $2\delta + 2$, then G has a rainbow matching of size δ.*

2. Proof of main result

Firstly, when $\delta = 1$ and $\delta = 2$, the proof is trivial. So we assume that $\delta \geq 3$. We prove it by contradiction. If Theorem 1.1 is false, then there exists a minimal δ such that there is no rainbow matching of size δ in G. By the minimality of δ, G has a rainbow matching of size $\delta - 1$. Suppose that M is a rainbow matching of size $\delta - 1$ in G. Let $c(e)$ denote the color of an edge e and $C(H)$ denote the color set of H, where H is a subgraph of G. We call a color *new*, if it is not in $C(M)$. Moreover, an edge with a new color is called a *new edge*. Let T denote the subgraph induced by $V(G) - V(M)$. Note that $C(T) \subseteq C(M)$ otherwise we can get a rainbow matching of size δ. For a vertex v in T, let $d_T(v)$ denote the degree of v in T and $d_N(v)$ denote the number of new edges incident with v in G. For a vertex $v \in V(M)$, let $d_N(v) = |\{vu \mid u \in V(T), c(vu) \notin C(M)\}|$. A *good* triangle $T_M(v, xy)$ is a triangle $vxyv$ such that $v \notin V(M)$, $xy \in E(M)$ and $c(vx), c(vy) \notin C(M)$.

Claim 2.1. *Given a maximum rainbow matching M and any vertex v not in $V(M)$, there exists a good triangle $T_M(v, e)$, where $e \in E(M)$.*

Proof. We prove it by contradiction. Recall that if vx is a new edge, then $x \in V(M)$. Let vx be a new edge incident with v and $xy \in E(M)$. Since G is strongly edge-colored, v cannot be incident with an edge-colored by $c(xy)$. Suppose that there exists no good triangle $T_M(v, e)$. There are $d_N(v)$ new edges incident with v, so there are at least $d_N(v)$ edges with colors in $C(M)$ cannot be incident with v. Thus the number of edges with colors in $C(M)$ and incident with v is at most $\delta - 1 - d_N(v)$. It follows that $d(v) \leq d_N(v) + \delta - 1 - d_N(v) = \delta - 1 < d(v)$, which is a contradiction. □

Let $M = \{x_1y_1, x_2y_2, \dots, x_{\delta-1}y_{\delta-1}\}$ and $V(T) = \{v_1, v_2, \dots, v_t\}$. Since $n \geq 2\delta + 2$, it follows that $t \geq 4$.

Claim 2.2. *For each edge $x_iy_i \in E(M)$, if $d_N(x_i) + d_N(y_i) \geq 3$, then $d_N(x_i) \times d_N(y_i) = 0$.*

Proof. Otherwise, we can choose two new edges x_iv and y_iu such that $v, u \in V(T)$ and $v \neq u$. Since G is strongly edge-colored, $c(x_iv) \neq c(y_iu)$. Hence we get a rainbow matching $M \cup \{x_iv, y_iu\} - x_iy_i$ of size δ, which is a contradiction. □

By Claim 2.1, each vertex v_i has a good triangle $T_M(v_i, xy)$. Relabeling the edges of M, we can assume that the good triangles are $T_M(v_1, x_1y_1), \dots, T_M(v_t, x_ty_t)$. (Recall that these good triangles are vertex-disjoint by Claim 2.2.) Let $M_1 = \{x_1y_1, \dots, x_ty_t\}$ and $M_2 = M - M_1$. Let x_iy_i be an edge of M_2. If $d_N(x_i) + d_N(y_i) \geq 3$, then we call x_iy_i a *nice* edge; (Note that $d_N(x_i) \times d_N(y_i) = 0$ by Claim 2.2, without loss of generality, we assume that $d_N(x_i) = 0$.) if $1 \leq d_N(x_i) + d_N(y_i) \leq 2$, then we call x_iy_i a *good* edge; otherwise, we call it a *bad* edge; if $c(x_iy_i) \in C(T)$, then we call it an *old* edge.

Claim 2.3. *An old edge must be a bad edge.*

Proof. Suppose our claim does not hold. Let x_iy_i be an old but not bad edge. Then it should be adjacent to a new edge, without loss of generality, let y_iw be a new edge where $w \in V(T)$. In addition, there is an edge $e \in E(T)$ such that $c(x_iy_i) = c(e)$. Recall that G is strongly edge-colored, e is not incident with w. So we get a rainbow matching $M \cup \{e, y_iw\} - x_iy_i$ of size δ, which is a contradiction. □

Claim 2.4. *If $v \in V(T)$, then $d_N(v) \geq \frac{\delta}{2} + 1 - \frac{d_T(v)}{2}$.*

Proof. Let $d_M(v)$ denote the number of edges vu such that $c(vu) \in C(M)$ and $u \in V(M)$. We know that $d(v) = d_N(v) + d_M(v) + d_T(v) \geq \delta$, so

$$d_N(v) \geq \delta - d_M(v) - d_T(v). \tag{2.1}$$

Since G is strongly edge-colored,

$$d_M(v) + d_T(v) \leq \delta - 1 - \frac{d_N(v) + d_M(v)}{2}.$$

It follows that

$$d_M(v) \leq \frac{2(\delta - 1)}{3} - \frac{d_N(v)}{3} - \frac{2d_T(v)}{3}. \tag{2.2}$$

By combining (2.1) and (2.2), we have that $d_N(v) \geq \frac{\delta}{2} + 1 - \frac{d_T(v)}{2}$. □