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S-packing colorings of cubic graphs

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ABSTRACT

Given a non-decreasing sequence $S = (s_1, s_2, \dots, s_k)$ of positive integers, an S-*packing coloring* of a graph G is a mapping c from $V(G)$ to $\{s_1, s_2, \dots, s_k\}$ such that any two vertices with the ith color are at mutual distance greater than s_i, $1 \le i \le k$. This paper studies S-packing colorings of (sub)cubic graphs. We prove that subcubic graphs are $(1, 2, 2, 2, 2, 2, 2)$-packing colorable and $(1, 1, 2, 2, 2)$-packing colorable. For subdivisions of subcubic graphs we derive sharper bounds, and we provide an example of a cubic graph of order 38 which is not $(1, 2, \dots, 12)$-packing colorable.

1. Introduction

A proper coloring of a graph G is a mapping which associates a color (integer) to each vertex such that adjacent vertices get distinct colors. In such a coloring, the color classes are stable sets (1-packings). As an extension, a d-distance coloring of G is a proper coloring of the dth power G^d of G, i.e. a partition of $V(G)$ into d-packings (sets of vertices at pairwise distance greater than d). While Brook's theorem implies that all cubic graphs except the complete graph K_4 of order 4 are properly 3-colorable, many authors studied 2-distance colorings of cubic graphs.

The aim of this paper is to study a mixing of these two types of colorings, i.e. colorings of (sub)cubic graphs in which some colors classes are 1-packings while other are d-packings, $d \ge 2$. Such colorings can be expressed using the notion of S-packing coloring. For a non-decreasing sequence $S = (s_1, s_2, \dots, s_k)$ of positive integers, an S-*packing coloring* (or simply S-coloring) of a graph G is a coloring of its vertices with colors from $\{s_1, s_2, \dots, s_k\}$ such that any two vertices with the ith color are at mutual distance greater than s_i, $1 \le i \le k$. The color class of each color s_i is thus an s_i-packing. The graph G is S-*colorable* if there exists an S-coloring and it is S-*chromatic* if it is S-colorable but not S'-colorable for any $S' = (s_1, s_2, \dots, s_j)$ with $j < k$ (notice that Goddard et al. [13] define differently the S-chromaticness for infinite graphs).

A $(d, \dots, d)$-coloring is thus a d-distance k-coloring, where k is the number of d (see [16] for a survey of results on this invariant) while a $(1, 2, \dots, d)$-coloring is a packing coloring. The packing chromatic number $\chi_\rho(G)$ of G is the integer k for which G is $(1, \dots, k)$-chromatic. This parameter was introduced by Goddard et al. [12] under the name of *broadcast chromatic number* and the authors showed that deciding whether $\chi_\rho(G) \le 4$ is NP-hard. A series of works [3,6,8,9,12,18] considered the packing chromatic number of infinite grids. For sequences S other than $(1, 2, \dots, k)$, S-packing colorings were considered more recently [11,14,13]. Other papers are about the complexity class of the decision problem associated to the S-packing coloring problem [7,10].

Regarding subcubic graphs, the packing chromatic number of the hexagonal lattice and of the infinite 3-regular tree is 7 and at most 7, respectively. Recently, Brešar et al. [4], have proven that the packing chromatic number of some cubic graphs, namely the base-3 Sierpiński graphs, is bounded by 9. Goddard et al. [12] asked what is the maximum of the

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Table 1
Number of S-chromatic cubic graphs of order n up to 22.

$n \setminus S$	(1, 2, 2, 2)	(1, 2, 2, 2, 2)	(1, 2, 2, 2, 2, 2)	(1, 2, 2, 2, 2, 2, 2)
4	1	0	0	0
6	1	1	0	0
8	2	1	2	0
10	11	7	0	1
12	11	74	0	0
14	254	250	5	0
16	1 031	3 017	12	0
18	15 960	25 297	44	0
20	178 193	332 045	251	0
22	2481 669	4835 964	1814	0

Table 2
Number of S-chromatic cubic graphs of order n up to 22.

$n \setminus S$	(1, 1)	(1, 1, 2)	(1, 1, 2, 3)	(1, 1, 2, 3, 3)
4	0	0	1	0
6	1	0	1	0
8	1	2	2	0
10	2	9	7	1
12	5	42	38	0
14	13	314	182	0
16	38	2 808	1 214	0
18	149	32 766	8 386	0
20	703	423 338	86 448	0
22	4132	6212 201	1103 114	0

Table 3
Number of cubic graphs of order n with packing chromatic number χ_ρ up to 20. *There are 55284 cubic graphs of order 20 and with packing chromatic number between 9 and 10 (our program takes too long time to compute their packing chromatic numbers).

$n \setminus \chi_\rho$	4	5	6	7	8	9	10	11
4	1	0	0	0	0	0	0	0
6	1	1	0	0	0	0	0	0
8	0	3	2	0	0	0	0	0
10	0	3	15	1	0	0	0	0
12	0	7	42	36	0	0	0	0
14	0	13	252	222	22	0	0	0
16	0	34	907	2 685	433	1	0	0
18	0	116	5 277	21 544	14 050	314	0	0
20	0	151	22 098	206 334	226 622	55284*		0

packing chromatic number of a cubic graph of order n. For 2-distance coloring of cubic graphs, Cranston and Kim have recently shown [5] that any subcubic graph is (2, 2, 2, 2, 2, 2, 2, 2)-colorable (they in fact proved a stronger statement for list coloring). For planar subcubic graphs G, there are also sharper results depending on the girth of G [2,5,15].

In this paper, we study S-packing colorings of subcubic graphs for various sequences S starting with one or two '1'. We also compute the distribution of S-chromatic cubic graphs up to 20 vertices, for three sequences S. The corresponding results are reported in Tables 1–3. They are obtained by an exhaustive search, using the lists of cubic graphs maintained by Gordon Royle [17]. The paper is organized as follows: Section 2 is devoted to the study of $(1, k, \dots, k)$-colorings of subcubic graphs for $k = 2$ or 3; Section 3 to $(1, 1, 2, \dots)$-colorings; Section 4 to $(1, 2, 3, \dots)$-colorings and Section 5 concludes the paper by listing some open problems.

1.1. Notation

To describe an S-coloring, if an integer s is repeated in the sequence S, then we will denote the colors s by $s_a, s_b, \dots$.

The *subdivided graph* $S(G)$ of a (multi)graph G is the graph obtained from G by subdividing each edge once, i.e. replacing each edge by a path of length two. In $S(G)$, vertices of G are called *original* vertices and other vertices are called *subdivision* vertices. Let us call a graph d-*irregular* if it has no adjacent vertices of degree d. Notice that graphs obtained from subcubic graphs by subdividing each edge at least once are 3-irregular graphs.

The following method (that is inspired from that of Cranston and Kim [5]) is used in the remainder of the paper to produce a desired coloring of a subcubic graph (except for Theorem 3): for a graph G and an edge $e = xy \in E(G)$, a *level ordering* of (G, e) is a partition of $V(G)$ into levels $L_i = \{v \in V(G) : d(v, e) = i\}$, $0 \leq i \leq \epsilon(e)$, with $\epsilon(e) = \max(\{d(u, e), u \in V(G)\}) \leq diam(G)$. The vertices are then colored one by one, from level $\epsilon(e)$ to 1, while preserving