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The sorting index and inversion number on order ideals of permutation groups

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ABSTRACT

Björner and Wachs showed that the major index and the inversion number are equidistributed on an order ideal U of the symmetric group in the weak order if and only if the maximal elements of U are 132-avoiding permutations. In this paper, we show that the sorting index and the inversion number are equidistributed on an order ideal U of the symmetric group in the Bruhat order if the maximal elements of U are 312-avoiding permutations. We also consider the case of type B. By introducing the notion of B-increasing subexcedent sequence, we show that the sorting index and inversion number are equidistributed on an order ideal U of the group of signed permutations in the Bruhat order if the A-code of each maximal element of U is B-increasing.

1. Introduction

Let S_n be the symmetric group on $\{1, 2, \ldots, n\}$. It is well known that the major index maj and the inversion number inv are two statistics with the same distribution on S_n. Foata [8] provided a bijective proof of this fact by constructing a bijection Φ on S_n such that $\mathrm{maj}(\pi) = \mathrm{inv}(\Phi(\pi))$ for any permutation $\pi \in S_n$. Björner and Wachs [3] showed that the bijection Φ can be used to find subsets U of S_n over which maj and inv are equidistributed. In particular, they proved that if U is an order ideal of S_n in the weak order, then $\Phi(U) = U$ if and only if the maximal elements of U are 132-avoiding permutations. Besides, invariant principal order ideals of S_n (in the Bruhat order other than the weak order) under the bijection Φ are characterized by Li and Miao [10].

Instead of major index, this paper is concerned with the equidistribution of another statistic—the sorting index, and the inversion number on order ideals of the group of permutations and signed permutations in the Bruhat order. The sorting index sor was introduced by Petersen [11], who showed that sor and inv have the same distribution on S_n algebraically. And then Chen, Gong and Guo [5] gave a combinatorial proof of this equidistribution by applying the bijection

$$\phi = \text{B-code}^{-1} \circ \text{A-code}$$

of Foata and Han [9]. It was shown by Poznanović [12] that the bijection ϕ has the property

$$(\mathrm{inv}, \mathrm{Rmil}, \mathrm{Lmal}, \mathrm{Lmap})\sigma = (\mathrm{sor}, \mathrm{Cyc}, \mathrm{Lmal}, \mathrm{Lmap})\phi(\sigma)$$

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for each $\sigma \in S_n$. Definitions of these statistics on S_n will be given in Section 2.1. By using the bijection ϕ, Poznanović proved that the set-valued statistics (inv, Rmil, Lmal, Lmap) and (sor, Cyc, Lmal, Lmap) are equidistributed on some restricted permutations.

We observe that the bijection ϕ plays a similar role, as the map Φ does in the work of Björner and Wachs [3], in the study of finding subsets of S_n on which inv and sor are equidistributed. More precisely, let U be an order ideal of S_n in the Bruhat order. We show that $\phi(U) = U$ if each maximal element of U is a 312-avoiding permutation. Then (sor, Cyc, Lmal, Lmap) and (inv, Rmil, Lmal, Lmap) are equidistributed on such order ideals U. We also compute the generating function of (sor, Cyc) or (inv, Rmil) on the principal order ideal Λ_π generated by a 312-avoiding permutation π.

As for the case of type B, let B_n denote the group of signed permutations of length n. The sorting index sor_B and the inversion number inv_B were shown to have the same distribution on B_n, see Petersen [11]. By introducing the A-code and B-code for B_n, Chen, Gong and Guo [5] defined a type B analogue ϕ_B of the bijection ϕ. Poznanović [12] showed that ϕ_B satisfies

$$(\mathrm{inv}_B, \mathrm{Rmil}_B, \mathrm{Lmal}_B, \mathrm{Lmap}_B)\sigma = (\mathrm{sor}_B, \mathrm{Cyc}_B, \mathrm{Lmal}_B, \mathrm{Lmap}_B)\phi_B(\sigma)$$

for each $\sigma \in B_n$. For definitions of these statistics on B_n, see Section 3.1.

To present the results for signed permutations, we introduce the notion of B-increasing subexcedent sequence. We show that, for an order ideal U of B_n in the Bruhat order, if the A-code of every maximal element of U is B-increasing, then $\phi_B(U) = U$. Thus $(\mathrm{sor}_B, \mathrm{Cyc}_B, \mathrm{Lmal}_B, \mathrm{Lmap}_B)$ and $(\mathrm{inv}_B, \mathrm{Rmil}_B, \mathrm{Lmal}_B, \mathrm{Lmap}_B)$ are equidistributed on U. We further give a formula for the generating function of the bistatistics $(\mathrm{sor}_B, \mathrm{Cyc}_B)$ or $(\mathrm{inv}_B, \mathrm{Rmil}_B)$ on principal order ideals of B_n generated by signed permutations with B-increasing A-code.

It is surprising to find that the number of signed permutations in B_n with B-increasing A-code is

$$C(n) + C(n+1) - 1,$$

which is also the number of fully commutative bottom elements of B_n introduced by Stembridge [14]. The fully commutative bottom elements of B_n can also be characterized by pattern avoidance of signed permutations, see [14].

2. The group of permutations

In this section, we consider the equidistribution of sor and inv on order ideals U of S_n in the Bruhat order. We show that if the maximal elements of U are 312-avoiding, then sor and inv have the same distribution on U.

2.1. Notation and terminology

This subsection reviews definitions of some permutation statistics, Bruhat order on the symmetric group and two permutation codes introduced by Foata and Han [9].

A permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in S_n$ is regarded as a one-to-one function on $\{1, \ldots, n\}$ such that $\sigma(i) = \sigma_i$ for $1 \le i \le n$. Multiplications of permutations are taken from right to left as composition of functions. That is, for $\sigma, \pi \in S_n$, we have $\sigma\pi(i) = \sigma(\pi(i))$ for $1 \le i \le n$. A permutation can be decomposed into disjoint cycles. Denote by (i, j) the transposition which interchanges i and j. Hence in particular, $\sigma(i, j)$ is the permutation obtained from σ by interchanging σ_i and σ_j.

A permutation σ can be uniquely decomposed as a product of transpositions

$$\sigma = (i_1, j_1)(i_2, j_2)\cdots(i_k, j_k),$$

where

$$i_r < j_r \quad \text{for } 1 \le r \le k$$

and

$$j_1 < j_2 < \cdots < j_k.$$

The sorting index sor of σ is defined by

$$\mathrm{sor}(\sigma) = \sum_{r=1}^{k}(j_r - i_r).$$

For a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n$, the inversion number $\mathrm{inv}(\sigma)$ and the major index $\mathrm{maj}(\sigma)$ of σ are defined by $\mathrm{inv}(\sigma) = |\{(\sigma_i, \sigma_j) : i < j, \sigma_i > \sigma_j\}|$ and $\mathrm{maj}(\sigma) = \sum_{i:\sigma_i>\sigma_{i+1}} i$, respectively. An element σ_i is called a left-to-right maximum letter of σ if $\sigma_j < \sigma_i$ for $j < i$, and the index i is called a left-to-right maximum position. Denote by $\mathrm{Lmal}(\sigma)$ (resp. $\mathrm{Lmap}(\sigma)$) the set of left-to-right maximum letters (resp. positions) of σ. Similarly, the set of right-to-left minimum letters of σ is denoted by $\mathrm{Rmil}(\sigma)$, and let $\mathrm{rmil}(\sigma) = |\mathrm{Rmil}(\sigma)|$. When σ is converted into disjoint cycles, denote by $\mathrm{Cyc}(\sigma)$ the set of minimal elements of each cycle, and let $\mathrm{cyc}(\sigma) = |\mathrm{Cyc}(\sigma)|$. The permutation σ is said to be 312-avoiding (resp. 132-avoiding) if there does not exist $i < j < k$ such that $\sigma_j < \sigma_k < \sigma_i$ (resp. $\sigma_i < \sigma_k < \sigma_j$). It is well known that the number of 312-avoiding (or 132-avoiding) permutations in S_n is the nth Catalan number $C(n)$.