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Note

Chambers of wiring diagrams

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ABSTRACT

Given a wiring diagram (pseudo-line arrangement) of a permutation $w \in S_n$, the chambers can be labeled with subsets of $[n]$ and they are called chamber sets. In this short note, we show that two wiring diagrams (of same w), can be mutated from one to another via the usual moves (coming from nil and braid relations), while freezing the chamber sets they have in common.

1. Introduction

A ***wiring diagram*** [4] of a permutation $w \in S_n$ is a planar configuration of n pseudo-lines (wires) $L_1, \ldots, L_n$ between two columns of $[n] := \{1, \ldots, n\}$ which satisfies:

- The wire L_i starts at $w^{-1}(i)$ and ends at i.
- No two wires intersect more than once.
- No three wires intersect at a point.

Each crossing depicts a ***simple transposition*** in S_n. In particular, if there are $i-1$ lines above the crossing, then the crossing corresponds to $s_i = (i, i+1)$, which is a simple transposition exchanging the ith and $i+1$th element. The number of crossings depends only on the permutation, is called the ***length*** of the permutation and is represented by the symbol $l(w)$. By reading off the simple transpositions from the crossings going from left to right, one can get a ***reduced word*** of the permutation, which is the minimal length representative of the permutation.

There are two types of relationships between the simple transpositions.

- (nil) $s_i s_j = s_j s_i$ for $|i-j| \geq 2$.
- (braid) $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ for $1 \leq i \leq n-1$.

We think of them as ***moves*** on the wiring diagram, since we can use these relations to change a reduced word of w to get another reduced word of w [2]. Fig. 1 shows an example of a braid move.

We can label the ***chambers***, the connected components of the complement of the union of all wires, by the following rule : the label of a chamber contains j if and only if the chamber lies above the wire ending at j. The labels of chambers, called ***chamber sets***, are used often to describe a certain set of minors. They were used to study quasi-commutativity between

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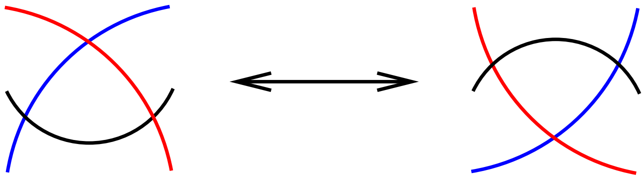

Fig. 1. A braid move involving blue, black and red wires. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

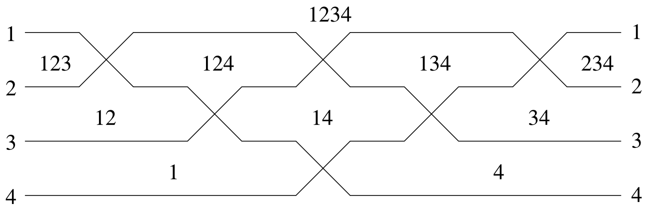

Fig. 2. A wiring diagram corresponding to $s_1s_2s_3s_1s_2s_1$.

quantum Plücker coordinates in [6] and they were used to study totally positive matrices in [1]. We will denote the collection of chamber sets for all chambers of a wiring diagram as the ***chamber collection*** of the diagram (see Fig. 2).

The main result of our paper is to show that:

Theorem 1.1. *Let w be a permutation of S_n and let $\mathcal{C}$ be a collection of subsets of $[n]$. Let $\mathcal{W}$ and $\mathcal{W}'$ be wiring diagrams of w whose chamber collections both contain $\mathcal{C}$. Then we can transform $\mathcal{W}$–$\mathcal{W}'$ using moves, preserving the chamber sets of $\mathcal{C}$.*

In other words, given a wiring diagram $\mathcal{W}$, pick some arbitrary collection of chambers. If that set of chamber labels appears on another wiring diagram $\mathcal{W}'$ of the same permutation, we can mutate $\mathcal{W}$–$\mathcal{W}'$ using moves, while freezing the chosen collection of chamber labels. This result also implies that the simplicial complex associated to chambers of wiring diagrams has a nice topological property (Section 3).

Remark 1.2. Throughout the paper we use the following notation. If S is a subset of $[n]$ and a an element of $[n]$, we may abbreviate $S \cup \{a\}$ and $S \setminus \{a\}$ by Sa and $S \setminus a$. In this paper, we need to deal with three levels of objects: elements of $[n]$, subsets of $[n]$, and collections of subsets of $[n]$. For clarity, we will denote these by lower case letters, capital letters, and calligraphic letters, respectively.

2. Proof of the main result

In this section, we will prove our main result, Theorem 1.1. Let $\mathcal{C}$ be a collection of subsets of $[n]$ and $\mathcal{W}_1$ and $\mathcal{W}_2$ be wiring diagrams of $w \in S_n$ such that the corresponding chamber collections both contain $\mathcal{C}$. We will say that $\mathcal{W}_1$ and $\mathcal{W}_2$ are $(w, \mathcal{C})$-equivalent if we can transform $\mathcal{W}_1$–$\mathcal{W}_2$ via moves while preserving the chambers of $\mathcal{C}$. Similarly, given two reduced words of w, we will say that they are $(w, \mathcal{C})$ equivalent if the corresponding chamber collections are.

When $\mathcal{C}$ is empty, the result is already well-known.

Lemma 2.1 (*Chapter 6 of [3]*). *Any two wiring diagrams of the same permutation are connected by a sequence of moves.*

A set $I \subseteq [n]$ is called a w-***chamber set*** if for each $j \in I$, the set I also contains all indices i such that $i < j$ and $w^{-1}(i) < w^{-1}(j)$. The meaning of this condition is the following : if L_i is above L_j and they do not cross, any chamber set containing i should also contain j. The w-***chamber sets*** $\mathcal{C}(w)$ stands for the collection of all $I \subseteq [n]$ such that I is a w-chamber set.

The following lemma strongly suggests we should use induction on $l(w)$ to prove the main result.

Lemma 2.2. *Let $w = w's_i$ be a permutation and $\mathcal{W}_1, \mathcal{W}_2$ be wiring diagrams of w such that the rightmost crossing is s_i for both of them. Let $\mathcal{W}_1'$ (respectively, $\mathcal{W}_2'$) denote the wiring diagram of w' obtained by deleting that rightmost crossing from $\mathcal{W}_1$ (respectively, $\mathcal{W}_2$). Then for each $\mathcal{C} \subset \mathcal{C}(w)$, there exists $\mathcal{C}' \subset \mathcal{C}(w')$ such that $\mathcal{W}_1'$ and $\mathcal{W}_2'$ being $(w', \mathcal{C}')$-equivalent implies $\mathcal{W}_1$ and $\mathcal{W}_2$ being $(w, \mathcal{C})$-equivalent.*

Proof. Construct $\mathcal{C}'$ from $\mathcal{C}$ by changing i to $i + 1$ for all sets that contain i but not $i + 1$. Then for any wiring diagram of w' whose chamber collection contains $\mathcal{C}'$, if we add the crossing s_i at the rightmost position, we get a wiring diagram of w whose chamber collection contains $\mathcal{C}$. □