Discrete Mathematics 339 (2016) 2515–2525

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

$(1, j)$-set problem in graphs

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ARTICLE INFO

Article history:
Received 9 December 2014
Received in revised form 4 April 2016
Accepted 5 April 2016
Available online 19 May 2016

Keywords:
Domination
$(1, j)$-set
NP-completeness
Probabilistic method
Chordal graph

ABSTRACT

A subset $D \subseteq V$ of a graph $G = (V, E)$ is a $(1, j)$-*set* (Chellali et al., 2013) if every vertex $v \in V \setminus D$ is adjacent to at least 1 but not more than j vertices in D. The cardinality of a minimum $(1, j)$-set of G, denoted as $\gamma_{(1,j)}(G)$, is called the $(1, j)$-*domination number* of G. In this paper, using probabilistic methods, we obtain an upper bound on $\gamma_{(1,j)}(G)$ for $j \geq O(\log \Delta)$, where Δ is the maximum degree of the graph. The proof of this upper bound yields a randomized linear time algorithm. We show that the associated decision problem is NP-complete for choral graphs but, answering a question of Chellali et al., provide a linear-time algorithm for trees for a fixed j. Apart from this, we design a polynomial time algorithm for finding $\gamma_{(1,j)}(G)$ for a fixed j in a split graph, and show that $(1, j)$-set problem is fixed parameter tractable in bounded genus graphs and bounded treewidth graphs.

1. Introduction

Domination and its variants have been the most active areas of research in graph theory. The concept of (i, j)-set is a very interesting and recent variant of domination [6,8,19].

1.1. Definitions and notations

For a natural number m, let $[m]$ denote the set $\{1, 2, \ldots, m\}$. Let $G = (V, E)$ be a finite graph. For $v \in V$, let $N_G(v) = \{u | uv \in E\}$ denote the open neighborhood of v and $N_G[v] = N_G(v) \cup \{v\}$ denote the closed neighborhood of v. The degree of a vertex $v \in V$, is denoted as $d_G(v)$. Let Δ_G and δ_G denote the maximum and minimum degree of G. (We will remove the subscript G where it is obvious from the context). Let $G[S]$ denote the subgraph induced by the vertex set S on G.

A vertex $u \in V$ is said to be dominated by a vertex $v \in V$ if $u \in N_G[v]$. A set $D \subseteq V$ is called a *dominating set* of G if for every vertex $v \in V \setminus D$, $|N_G(v) \cap D| \geq 1$. The cardinality of a minimum dominating set of G is called the *domination number* of G and is denoted by $\gamma(G)$. Note that, a dominating set D dominates each vertex of $V \setminus D$ at least once. If, for some positive integer i, a dominating set D_i dominates each vertex of $V \setminus D_i$ at least i times, then D_i is called a i-dominating set. A *restrained dominating set* is a set $D_r \subseteq V$ where every vertex in $V \setminus D_r$ is adjacent to a vertex in D_r as well as another vertex in $V \setminus D_r$. The cardinality of a minimum restrained dominating set of G is called the *restrained domination number* of G.

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http://dx.doi.org/10.1016/j.disc.2016.04.008

1.2. Short review on (i, j)-set

A set $D \subseteq V$ of a graph $G = (V, E)$ is called a (i, j)-*set* if for every $v \in V \setminus D$, $i \leq |N_G(v) \cap D| \leq j$ for nonnegative integers i and j, that is, every vertex $v \in V \setminus D$ is adjacent to at least i but not more than j vertices in D. The decision version of (i, j)-set problem is defined as follows.

(i, j)-**Set problem** ((i, j)-**SET**)

Instance: A graph $G = (V, E)$ and a positive integer $k \leq |V|$.
Question: Does there exist a (i, j)-set D of G such that $|D| \leq k$?

The concept of (i, j)-set was introduced by Chellali et al. in [6]. In an earlier work, Dejter [8] defined *quasiperfect domination*—a set D is called a quasiperfect dominating set if all vertices in $V \setminus D$ are adjacent to either one or two vertices of D. By definition, a quasiperfect dominating set is basically a $(1, 2)$-set. Clearly, both are generalizations of the classical domination problem. Like domination problem, in this case, our goal is to find a (i, j)-set of minimum cardinality, which is called the (i, j)-*domination number* of G and is denoted by $\gamma_{(i,j)}(G)$. Basically, we are interested in finding a i-dominating set with a bounded redundancy. In these type of situations, we need the concept of (i, j)-set. Also, (i, j)-set is a more general concept which involves *nearly perfect set* [9], *perfect dominating set* [15] (also known as 1-*fair dominating set* [5]) etc. as variants. There is a concept of *set restricted domination* [2] which is defined as follows: for each vertex $v \in V$, we assign a set S_v. A set D_S is called a set restricted dominating set if for all $v \in V$, $|N_G[v] \cap D_S| \in S_v$. Note that if $S_v = [j]$ for all $v \in V$, we have a $(1, j)$-set. In that sense, $(1, j)$-set is a particular type of set restricted dominating set.

The concept of (i, j)-set, whose origin can be traced back to quasiperfect domination [8], has been introduced recently in 2013 [6]. To the best of our knowledge, only three papers have appeared on (i, j)-set or its variants [6,19,8]. The main focus of [6] is on a particular (i, j)-set, namely $(1, 2)$-set. In [6], the authors observed that for a simple graph G with n vertices, $\gamma(G) \leq \gamma_{(1,2)}(G) \leq n$. They showed that $\gamma(G) = \gamma_{(1,2)}(G)$ for claw-free graphs, P_4-free graphs, caterpillars, etc. The authors have constructed a special type of split graph that achieves the upper bound. But they also showed that there are some graph classes for which $\gamma_{(1,2)}(G)$ is strictly less than n. The authors have also studied the $(1, 3)$-set for grid graphs and showed that $\gamma(G) = \gamma_{(1,3)}(G)$. Using this result, they also showed that domination number is equal to restrained domination number. From complexity point of view, it is known that $(1, 2)$-SET is NP-complete for bipartite graphs [6]. A list of open problems were posed in [6], some of which were solved in [19]. In [19], the authors showed that there exist planar and bipartite graphs with $\gamma_{(1,2)}(G) = n$. They also showed that for a tree T with k leaves, if $deg_G(v) \geq 4$ for any non-leaf vertex v, then $\gamma_{(1,2)}(T) = n - k$. Nordhaus–Gaddum-type inequalities are also established for $(1, 2)$-set in [19]. In [8], quasiperfect domination has been studied in triangular lattices.

The main focus of [6] and [19] is $(1, 2)$-set. In this paper, we study the more general set, namely $(1, j)$-set. Apart from the open problems mentioned in [6], a bound on the $(1, j)$-domination number for general graphs is important. A general bound for sufficiently large j (approximately, j is at least $O(\log \Delta)$) and its construction forms a major thrust of this paper, which is presented in Section 2. In Section 3, we tighten the hardness result by showing that $(1, j)$-set problem is NP-complete for chordal graphs. Section 4 focuses on algorithms for $(1, j)$-set problem. We propose a linear time algorithm for finding $\gamma_{(1,j)}(G)$ for a tree, a polynomial time algorithm for finding $\gamma_{(1,j)}(G)$ for a fixed j in a split graph, and fixed parameter tractable results for bounded genus graph and bounded treewidth graphs. Finally, Section 5 concludes the paper.

2. Upper bounds

In this section, we shall prove an upper bound on the $(1, j)$-domination number, i.e. $\gamma_{(1,j)}(G)$, of any graph $G = (V, E)$, having minimum degree, δ and maximum degree, Δ, for *sufficiently large* j. Here the asymptotics are for n and Δ going to infinity *independently of each other*. Our result holds when j is sufficiently large as a function of Δ ($j \geq O(\log \Delta)$).

In [1], Alon and Spencer describe a similar upper bound on the domination number $\gamma(G)$, using probabilistic methods. Their strategy, a classic example of the 'alteration technique', was to select a random subset X of vertices as a partial dominating set, and then to include the set Y of vertices not dominated by X, to get the final dominating set. However, such a strategy is *a priori* not applicable for $(1, j)$-domination, because including or excluding vertices from the dominating set could change the number of dominating vertices adjacent to some vertex. Instead, we shall use a one-step process, and analyze it using the Lovász Local Lemma and Chernoff bounds, to ensure that the conditions for $(1, j)$-set hold. To use LLL, we need to bound the probability of the bad event, which is the event that a vertex is dominated more than j times. Intuitively, as j increases, this probability goes down. For this reason, the bound proved in this section is valid for large j, namely $j \geq O(\log \Delta)$. Our proof also implies a randomized linear time algorithm, using the Moser–Tardos constructive version of the Local Lemma [17].

We first state two well-known results, Chernoff bound and Lovász Local Lemma, in a form suitable for our purposes. These results can be found in any standard text on probabilistic combinatorics, e.g. [1].

Theorem 1 (*Chernoff Bound*). *Suppose X is the sum of n independent random variables, each equal to 1 with probability p and 0 otherwise. Then for any $\alpha \geq 0$, $\Pr(X > (1 + \alpha)np) < \exp(-f(\alpha)np)$, where $f(\alpha) = (1 + \alpha)\ln(1 + \alpha) - \alpha$.*