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New bounds for the acyclic chromatic index

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ABSTRACT

An edge coloring of a graph G is called an *acyclic edge coloring* if it is proper and every cycle in G contains edges of at least three different colors. The least number of colors needed for an acyclic edge coloring of G is called the *acyclic chromatic index* of G and is denoted by $a'(G)$. Fiamčik (1978) and independently Alon, Sudakov, and Zaks (2001) conjectured that $a'(G) \leq \Delta(G)+2$, where $\Delta(G)$ denotes the maximum degree of G. The best known general bound is $a'(G) \leq 4(\Delta(G)-1)$ due to Esperet and Parreau (2013). We apply a generalization of the Lovász Local Lemma to show that if G contains no copy of a given bipartite graph H, then $a'(G) \leq 3\Delta(G) + o(\Delta(G))$. Moreover, for every $\varepsilon > 0$, there exists a constant c such that if $g(G) \geq c$, then $a'(G) \leq (2+\varepsilon)\Delta(G) + o(\Delta(G))$, where $g(G)$ denotes the girth of G.

1. Introduction

All graphs considered here, unless indicated otherwise, are finite, undirected, and simple. An edge coloring of a graph G is called an *acyclic edge coloring* if it is proper (i.e., adjacent edges receive different colors) and every cycle in G contains edges of at least three different colors (there are no *bichromatic cycles* in G). The least number of colors needed for an acyclic edge coloring of G is called the *acyclic chromatic index* of G and is denoted by $a'(G)$. The notion of acyclic (vertex) coloring was first introduced by Grünbaum [8]. The edge version was first considered by Fiamčik [7], and independently by Alon, McDiarmid, and Reed [1].

As for many other graph parameters, it is quite natural to ask for an upper bound on the acyclic chromatic index of a graph G in terms of its maximum degree $\Delta(G)$. Since $a'(G) \geq \chi'(G) \geq \Delta(G)$, where $\chi'(G)$ denotes the ordinary chromatic index of G, this bound must be at least linear in $\Delta(G)$. The first linear bound was given by Alon et al. [1], who showed that $a'(G) \leq 64\Delta(G)$. Although it resolved the problem of determining the order of growth of $a'(G)$ in terms of $\Delta(G)$, it was conjectured that the sharp bound should be lower.

Conjecture 1 (*Fiamčik [7]; Alon, Sudakov, Zaks [2]*). *For every graph G, $a'(G) \leq \Delta(G)+2$.*

Note that the bound in Conjecture 1 is only one more than Vizing's bound on the chromatic index of G. However, this elegant conjecture is still far from being proven.

The first major improvement to the bound $a'(G) \leq 64\Delta(G)$ was made by Molloy and Reed [11], who proved that $a'(G) \leq 16\Delta(G)$. This bound remained the best for a while, until Ndreca, Procacci, and Scoppola [14] managed to improve it to $a'(G) \leq \lceil 9.62(\Delta(G)-1)\rceil$. This estimate was recently lowered further to $a'(G) \leq 4(\Delta(G)-1)$ by Esperet and Parreau [6].

All the bounds mentioned above were derived using probabilistic arguments, and recent progress was stimulated by discovering more sophisticated and powerful analogs of the Lovász Local Lemma, namely the stronger version of the LLL due to Bissacot, Fernández, Procacci, and Scoppola [4] and the entropy compression method of Moser and Tardos [12].

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The probability that a cycle would become bichromatic in a random coloring is less if the cycle is longer. Thus, it should be easier to establish better bounds on the acyclic chromatic index for graphs with high enough girth. Indeed, Alon et al. [2] showed that if $g(G) \geq c_1 \Delta(G) \log \Delta(G)$, where c_1 is some universal constant, then $a'(G) \leq \Delta(G) + 2$. They also proved that if $g(G) \geq c_2 \log \Delta(G)$, then $a'(G) \leq 2\Delta(G) + 2$. This was lately improved by Muthu, Narayanan, and Subramanian [13] in the following way: For every $\varepsilon > 0$, there exists a constant c such that if $g(G) \geq c \log \Delta(G)$, then $a'(G) \leq (1+\varepsilon)\Delta(G) + o(\Delta(G))$.

We are now turning to the case when $g(G)$ is bounded below by some constant independent of $\Delta(G)$, which will be the main topic of this paper. The first bounds of such type were given by Muthu et al. [13], who proved that $a'(G) \leq 9\Delta(G)$ if $g(G) \geq 9$, and $a'(G) \leq 4.52\Delta(G)$ if $g(G) \geq 220$. Esperet and Parreau [6] not only improved both these estimates even in the case of arbitrary $g(G)$, but they also showed that $a'(G) \leq \lceil 3.74(\Delta(G) - 1) \rceil$ if $g(G) \geq 7$, $a'(G) \leq \lceil 3.14(\Delta(G) - 1) \rceil$ if $g(G) \geq 53$, and, in fact, for every $\varepsilon > 0$, there exists a constant c such that if $g(G) \geq c$, then $a'(G) \leq (3+\varepsilon)\Delta(G) + o(\Delta(G))$.

The result that we present here consists in further improvement of the latter bounds. Namely, we establish the following.

Theorem 2. *Let G be a graph with maximum degree Δ and let H be some bipartite graph. If G does not contain H as a subgraph, then $a'(G) \leq 3\Delta + o(\Delta)$.*

Remark 3. In our original version of Theorem 2 we considered only the case where H was the 4-cycle. That almost the same proof in fact works for any bipartite H was observed by Esperet and de Verclos.

Remark 4. The function $o(\Delta)$ in the statement of Theorem 2 depends on H. In fact, our proof shows that for the complete bipartite graph $K_{k,k}$, it is of the order $O\left(\Delta^{1-1/2k}\right)$.

Theorem 5. *For every $\varepsilon > 0$, there exists a constant c such that for every graph G with maximum degree Δ and $g(G) \geq c$, we have $a'(G) \leq (2+\varepsilon)\Delta + o(\Delta)$.*

Remark 6. The bound of the last theorem was recently improved further to $a'(G) \leq (1+\varepsilon)\Delta + o(\Delta)$ by Cai, Perarnau, Reed, and Watts [5] using a different (and much more sophisticated) argument.

To prove Theorems 2 and 5, we use a generalization of the Lovász Local Lemma that we call the *Local Cut Lemma* (the LCL for short). The LCL is inspired by recent combinatorial applications of the entropy compression method, although its proof is probabilistic and does not use entropy compression. Several examples of applying the LCL and its proof can be found in [3]. We provide all the required definitions and the statement of the LCL in Section 2. In Section 3 we prove Theorem 2, and in Section 4 we prove Theorem 5.

2. The Local Cut Lemma

Roughly speaking, the Local Cut Lemma asserts that if a random set of vertices in a directed graph is cut out by a set of edges which is "locally small", then this set of vertices has to be "large" with positive probability. To state it rigorously, we will need some definitions.

By a *digraph* we mean a finite directed multigraph. Suppose that D is a digraph with vertex set V and edge set E. For x, $y \in V$, let $E(x, y) \subseteq E$ denote the set of all edges with tail x and head y.

A digraph D is *simple* if for all $x, y \in V$, $|E(x, y)| \leq 1$. If D is simple and $|E(x, y)| = 1$, then the unique edge with tail x and head y is denoted by xy (or sometimes (x, y)). For an arbitrary digraph D, let D^s denote its *underlying simple digraph*, i.e., the simple digraph with vertex set V in which xy is an edge if and only if $E(x, y) \neq \emptyset$. Denote the edge set of D^s by E^s. For a set $F \subseteq E$, let $F^s \subseteq E^s$ be the set of all edges $xy \in E^s$ such that $F \cap E(x, y) \neq \emptyset$.

A set $A \subseteq V$ is *out-closed* (resp. *in-closed*) if for all $xy \in E^s$, $x \in A$ implies $y \in A$ (resp. $y \in A$ implies $x \in A$).

Definition 7. Let D be a digraph with vertex set V and edge set E and let $A \subseteq V$ be an out-closed set of vertices. A set $F \subseteq E$ of edges is an *A-cut* if A is in-closed in $D^s - F^s$.

In other words, an A-cut F has to contain at least one edge $e \in E(x, y)$ for each pair x, y such that $x \in V \setminus A$, $y \in A$, and $xy \in E^s$.

To understand the motivation behind the LCL, suppose that we are given a random out-closed set of vertices $A \subseteq V$ and a random A-cut $F \subseteq E$ in a digraph D. Since A is out-closed, for every $xy \in E^s$,

$$\Pr(x \in A) \leq \Pr(y \in A).$$

We would like to establish a similar inequality in the other direction. More precisely, we want to find a function $\omega : E^s \to \mathbb{R}_{\geq 1}$ such that for all $xy \in E^s$,

$$\Pr(y \in A) \leq \Pr(x \in A) \cdot \omega(xy). \tag{1}$$

Note that if we can prove (1) for some function ω, then for all $xy \in E^s$,

$$\Pr(x \in A) \geq \frac{\Pr(y \in A)}{\omega(xy)},$$