Discrete Mathematics 339 (2016) 2553–2556

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Note

An analogue of Franklin's Theorem

O.V. Borodin [a], A.O. Ivanova [b,*]

[a] Sobolev Institute of Mathematics, Novosibirsk, 630090, Russia

[b] Ammosov North-Eastern Federal University, Yakutsk, 677000, Russia

ARTICLE INFO

Article history:
Received 8 September 2015
Received in revised form 29 April 2016
Accepted 30 April 2016
Available online 25 May 2016

Keywords:
Planar graph
Plane map
Structure properties
3-polytope
Weight

ABSTRACT

Back in 1922, Franklin proved that every 3-polytope P_5 with minimum degree 5 has a 5-vertex adjacent to two vertices of degree at most 6, which is tight. This result has been extended and refined in several directions.

The purpose of this note is to prove that every P_5 has a vertex of degree at most 6 adjacent to a 5-vertex and another vertex of degree at most 6, which is also tight. Moreover, we prove that there is no tight description of 3-paths in P_5s other than these two.

1. Introduction

The *degree* $d(x)$ of a vertex or face x in a plane graph G is the number of its incident edges. A k-*vertex* (k-*face*) is a vertex (face) with degree k, a k^+-*vertex* has degree at least k, etc. The minimum vertex degree of G is $\delta(G)$. We will drop the arguments whenever this does not lead to confusion.

A k-*path* is a path on k vertices. A path uvw is an (i, j, k)-*path* if $d(u) \leq i$, $d(v) \leq j$, and $d(w) \leq k$. The *weight* $w(H)$ of a subgraph H of a graph G is the degree-sum of the vertices of H in G. By $\mathbf{P}_\delta$ denote the class of 3-polytopes with minimum degree δ; in particular, $\mathbf{P_3}$ is the set of all 3-polytopes.

In 1904, Wernicke [35] proved that if $P_5 \in \mathbf{P_5}$ then P_5 contains a 5-vertex adjacent to a 6^--vertex. This result was strengthened by Franklin [19] in 1922 by proving the existence of a $(6, 5, 6)$-path in every P_5.

Theorem 1 (*Franklin [19]*)**.** *Every 3-polytope with minimum degree 5 has a $(6, 5, 6)$-path, which is tight.*

We recall that a description of 3-paths is *tight* if none of its parameters can be strengthened and no term dropped. The tightness of Franklin's description is shown by putting a vertex inside each face of the dodecahedron and joining it to the five boundary vertices.

Franklin's Theorem 1 is fundamental in the structural theory of planar graphs; it has been extended or refined in several directions, see [1–18,20–24,26–34] and a survey Jendrol'–Voss [25].

We now mention only a few easily formulated results on $\mathbf{P_5}$, which are the closest to Franklin's Theorem and whose parameters are all sharp.

Borodin [3] proved that there is a 3-face with weight at most 17. Jendrol' and Madaras [23] ensured a 5-vertex that has three neighbors whose weight sums to at most 18 and a 4-path with weight at most 23. Madaras [29] found a 5-path with weight at most 29.

* Corresponding author.
E-mail addresses: brdnoleg@math.nsc.ru (O.V. Borodin), shmgnanna@mail.ru (A.O. Ivanova).

http://dx.doi.org/10.1016/j.disc.2016.04.019
0012-365X/

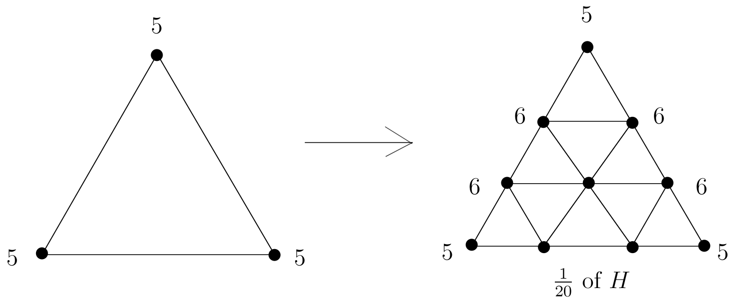

Fig. 1. A construction showing the tightness of Theorem 2.

All of a sudden, we have realized that the following fact, maybe the most similar to Franklin's Theorem, is missing in the literature.

Theorem 2. *Every* 3*-polytope with minimum degree* 5 *has a* (5, 6, 6)*-path, which is tight.*

Recently, we proved [11] that there exist precisely seven tight descriptions of 3-paths in triangle-free 3-polytopes.

Theorem 3 (*Borodin and Ivanova [11]*)**.** *There exist precisely seven tight descriptions of* 3*-paths in triangle-free* 3*-polytopes:*

(i) $(5, 3, 6) \vee (4, 3, 7)$,
(ii) $(3, 5, 3) \vee (3, 4, 4)$,
(iii) $(5, 3, 6) \vee (3, 4, 3)$,
(iv) $(3, 5, 3) \vee (4, 3, 4)$,
(v) $(5, 3, 7)$,
(vi) $(3, 5, 4)$,
(vii) $(5, 4, 6)$.

Problem 4 (*Borodin, Ivanova, and Kostochka [16]*)**.** Describe all tight descriptions of 3-paths in $\mathbf{P_3}$.

Another purpose of our short note is to make the following modest contribution to Problem 4.

Theorem 5. *There are no tight descriptions of* 3*-paths in* P_5 *s other than those given by Franklin's Theorem and Theorem* 2*.*

2. Proving Theorem 2

To show the tightness of Theorem 2, it suffices to replace each face of the icosahedron by the configuration shown in Fig. 1. Indeed, the resulting graph H_2 has neither (5, 6, 5)-paths nor (5, 5, 6)-paths.

Now suppose that a 3-polytope P'_5 contradicts Theorem 2 by avoiding (5, 6, 6)-paths. Let P_5 be a counterexample to Theorem 2 on the same vertices as P'_5 having the most edges.

Let $v_1, \dots, v_{d(x)}$ denote the neighbors of a vertex or a face x in a cyclic order round x.

(*) P_5 *is a triangulation.*

Indeed, suppose P_5 has a 4^+-face $f = v_1, \dots, v_{d(f)}$. If $d(v_1) \geq 6$ or $d(v_3) \geq 6$, then adding the diagonal $d = v_1v_3$ results in a counterexample P_5^* to Theorem 2 with more edges since d joins in P_5^* two 7^+-vertices, which contradicts the definition of P_5. Thus $d(v_1) = d(v_3) = 5$ in P_5. By symmetry, we have also $d(v_2) = d(v_4) = 5$. This means that P_5 has a (5, 5, 5)-paths, a contradiction.

Denote the sets of vertices, edges, and faces of P_5 by V, E and F, respectively. Euler's formula $|V| - |E| + |F| = 2$ for P_5 yields

$$\sum_{v \in V} (d(v) - 6) = -12. \tag{1}$$

We assign an *initial charge* $\mu(v) = d(v) - 6$ to each $v \in V$. Note that only 5-vertices have a negative initial charge.

Using the properties of G as a counterexample to Theorem 2, we will define a local redistribution of charges, preserving their sum, such that the *new charge* $\mu'(v)$ is non-negative whenever $v \in V$. This will contradict the fact that the sum of the new charges is, by (1), equal to −12, and this contradiction will finish the proof of Theorem 2.

Namely, we use the following discharging rules.

R1. *Every* 6^+*-vertex gives* $\frac{1}{4}$ *to every adjacent* 5*-vertex.*
R2. *Every* 7^+*-vertex gives* $\frac{1}{8}$ *to every adjacent* 6*-vertex.*