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The weight of faces in normal plane maps

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ABSTRACT

The weight of a face in a 3-polytope is the degree-sum of its incident vertices, and the weight of a 3-polytope, w, is the minimum weight of its faces. A face is pyramidal if it is either a 4-face incident with three 3-vertices, or a 3-face incident with two vertices of degree at most 4. If pyramidal faces are allowed, then w can be arbitrarily large, so we assume the absence of pyramidal faces in what follows.

In 1940, Lebesgue proved that every quadrangulated 3-polytope has $w \leq 21$. In 1995, this bound was lowered by Avgustinovich and Borodin to 20. Recently, we improved it to the sharp bound 18.

For plane triangulations without 4-vertices, Borodin (1992), confirming the Kotzig conjecture of 1979, proved that $w \leq 29$, which bound is sharp. Later, Borodin (1998) proved that $w \leq 29$ for all triangulated 3-polytopes. Recently, we obtained the sharp bound 20 for triangle-free polytopes.

In 1996, Horňák and Jendrol' proved for arbitrarily polytopes that $w \leq 32$. In this paper we improve this bound to 30 and construct a polytope with $w = 30$.

1. Introduction

By a 3-polytope we mean a finite convex 3-dimensional polytope. As proved by Steinitz [26], the 3-polytopes are in 1–1 correspondence with the 3-connected planar graphs.

A plane map is *normal* (NPM) if each of its vertices and faces is incident with at least three edges. Clearly, every 3-polytope is an NPM.

The *degree* $d(x)$ of a vertex or face x in an NPM M is the number of incident edges. A *k-vertex* or *k-face* is one of degree k, a k^+*-vertex* has degree at least k, a k^-*-face* has degree at most k, and so on.

The *weight* $w(f)$ of a face f in M is the degree-sum of its incident vertices. The *weight* $w(M)$ (or simply w) of a map M is the minimum weight of faces in M.

A 3-face is *pyramidal* if it is incident with at least two 4^--vertices, and a 4-face is *pyramidal* if it is incident with at least three 3-vertices.

If M has pyramidal faces, then w can be arbitrarily large. Indeed, every face f of the Archimedean $(3, 3, 3, n)$- and $(4, 4, n)$-solids satisfies $w(f) \geq n + 8$. Therefore, speaking about w in what follows, we assume that NPMs have no pyramidal faces

We now recall some results about the structure of 5^--faces in an NPM. By δ denote the minimum degree of vertices in M. We say that f is a *face of type* $(k_1, k_2, \ldots)$ or simply $(k_1, k_2, \ldots)$*-face* if the set of its incident vertices is majorized by the vector $(k_1, k_2, \ldots)$.

In 1940, Lebesgue [22] gave an approximated description of 5^--faces in NPMs.

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Theorem 1 (*Lebesgue [22]*)**.** *Every normal plane map has a* 5^-*-face of one of the following types:*

$$(3,6,\infty), (3,7,41), (3,8,23), (3,9,17), (3,10,14), (3,11,13),$$
$$(4,4,\infty), (4,5,19), (4,6,11), (4,7,9), (5,5,9), (5,6,7),$$
$$(3,3,3,\infty), (3,3,4,11), (3,3,5,7), (3,4,4,5), (3,3,3,3,5).$$

The classical Theorem 1, along with other ideas in Lebesgue [22], has numerous applications to coloring problems on plane graphs (first examples of such applications and a recent survey can be found in [7,24]). In 2002, Borodin [6] strengthened Theorem 1 in six parameters without worsening the others. However, the question in [6] of finding the best possible version(s) of Theorem 1 remains open, even for the special case of quadrangulations. Precise descriptions of 5^--faces are obtained for NPMs with $\delta = 5$ (Borodin [2]) and $\delta \geq 4$ (Borodin–Ivanova [8]), and also for triangulations (Borodin, Ivanova, Kostochka [11]).

Some parameters of Lebesgue's Theorem were improved for special classes of plane graphs. In 1989, Borodin [2] proved, confirming Kotzig's conjecture [20] of 1963, that every normal plane map with $\delta = 5$ has a $(5,5,7)$-face or $(5,6,6)$-face, where all parameters are best possible. This result also confirmed Grünbaum's conjecture [15] of 1975 that the cyclic connectivity (defined as the minimum number of edges to be deleted from a graph so as to obtain two components each of which has a cycle) of every 5-connected plane graph is at most 11, which bound is sharp (earlier, Plummer [25] obtained the bound 13).

For the plane triangulations without 4-vertices, Kotzig [21] proved that $w \leq 39$, and Borodin [4] proved, confirming Kotzig's conjecture [21], that $w \leq 29$; this bound is best possible, as follows from the construction obtained from the icosahedron by twice inserting a 3-vertex into each face. Borodin [5] further showed that $w \leq 29$ for every triangulated 3-polytope.

In 1940, Lebesgue [22] proved that every quadrangulated 3-polytope satisfies $w \leq 21$. In 1995, this bound was improved in Avgustinovich–Borodin [1] to 20. Recently, we [9] improved this bound to the sharp bound 18, and for triangle-free polytopes we [10] recently obtained the best possible bound 20.

Borodin and Woodall [12], with the additional assumption of the absence of $(3,5,\infty)$-faces, proved that there is either a $(3,6,20)$-face or a 5^--face of weight at most 25, which bound is sharp. We note that the weight of all 5^--faces can be at least 38 in the presence of $(3,5,\infty)$-faces (Horňák–Jendrol' [16]). On the other hand, Horňák and Jendrol' [16] proved that there is a 5^--face f with $w(f) \leq 47$.

Other results related to Lebesgue's Theorem can be found in the above mentioned papers and also in [18,13,3,14,17,19,23,27].

For arbitrary polytopes, Horňák and Jendrol' [16] (1996) proved that $w \leq 32$. The purpose of our paper is to improve this bound to $w \leq 30$ and construct a 3-polytope with $w = 30$.

Theorem 2. *Every normal plane map without pyramidal faces has a face of weight at most* 30*, which bound is sharp.*

Corollary 3. *Every* 3*-polytope without pyramidal faces has a face of weight at most* 30*, which bound is sharp.*

2. Proving Theorem 2

The bound 30 is attained at the construction in Fig. 1 derived from the $(3,3,3,3,5)$ Archimedean solid. Indeed, we put the inserts presented on the left in Fig. 1 into each face of the $(3,3,3,3,5)$ Archimedean solid, in which every vertex is incident with four triangles and a pentagon. As a result, we have a 3-polytope such that every 3-face is incident with a 22-vertex, and so it has weight at least 30, while each 10-face has weight 40.

Now let a normal plane map M' be a counterexample to the upper bound in Theorem 2. Starting from M', we construct a counterexample M with some useful properties.

The *operation* HD_{uw} consists in putting a diagonal uw into a 4^+-face f to split f into two non-pyramidal faces. For example, HD_{uw} works if $d(u) \geq 22$ and $d(w) \geq 4$, since it neither creates pyramidal faces, nor produces faces of weight less than $22 + 1 + 4 + 1 + 3 = 31$. Other possibilities of applying HD_{uw} will be given below.

We apply HD_{uw} to M' as many times as possible; this results in a counterexample M.

2.1. Basic structural properties of the counterexample M

(P1) *M has no* 6^+*-face incident with a* 20^+*-vertex u.*

Otherwise, we could apply HD_{uw}, where w is at distance at least three along the boundary of f, which splits f into two non-pyramidal faces with weight at least 31, contrary to the definition of M.

(P2) *M has no* 4^+*-face* $\ldots v_2v_1$ *with* $d(v_2) \geq 4$, $d(v_i) \geq 4$, *where* $i \notin \{1,3\}$ *such that* $d(v_2) + d(v_i) \geq 26$.